

THE GENERAL CHARACTERISTICS OF THE FREQUENCY FUNCTION OF STELLAR MOVEMENTS

AS DERIVED FROM THE PROPER MOTIONS
OF THE STARS

BY

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WITH 8 FIGURES IN TEXT

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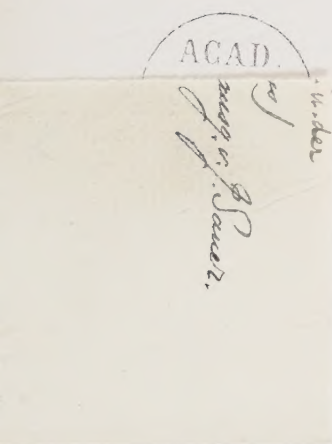


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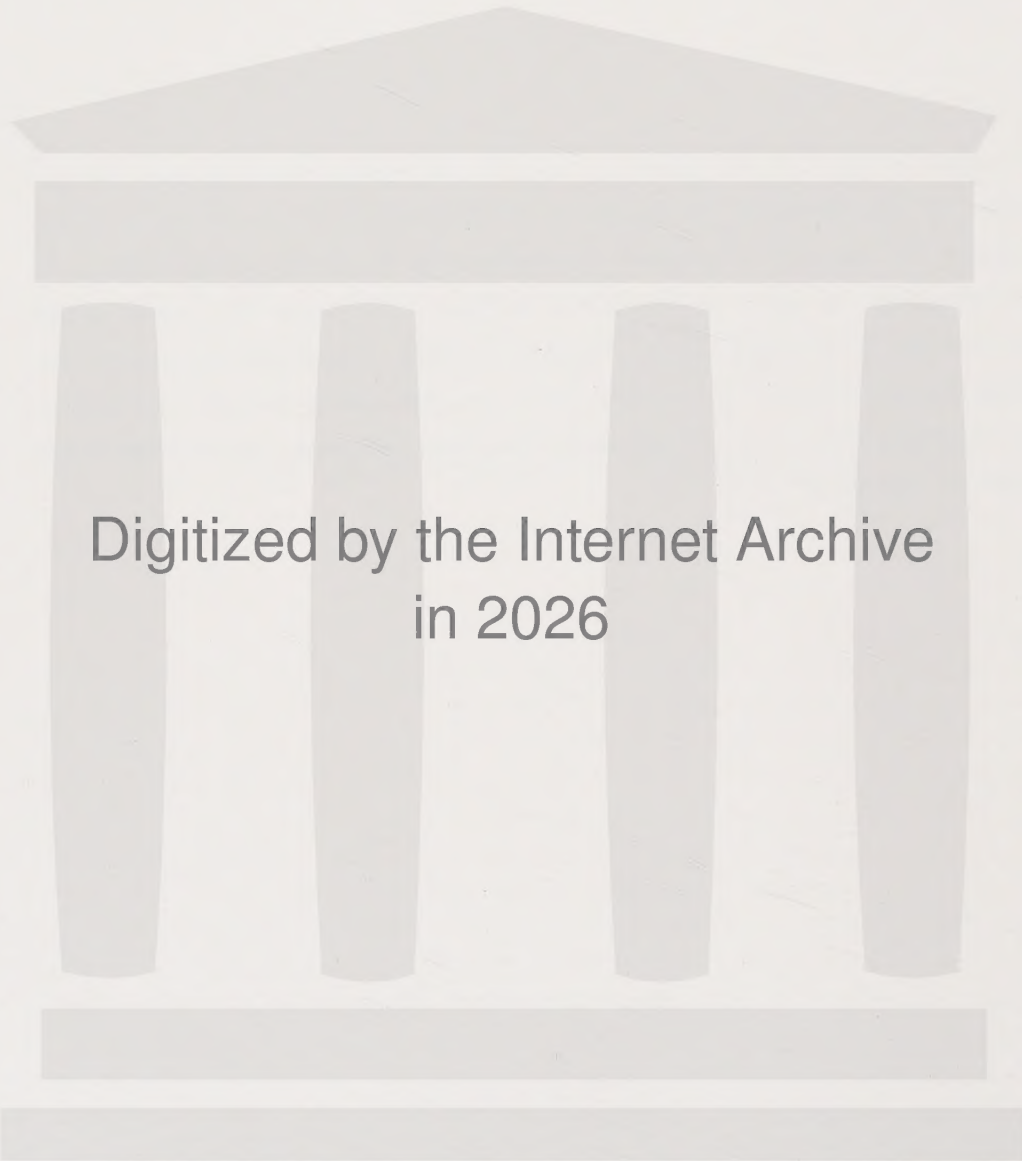


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This paper is one in the series of memoirs on Stellar Statistics that was inaugurated by the works of CHARLIER in Meddelanden N:s 8 and 9 of the Lund Observatory. It aims at a general characterisation of the frequency-function of stellar movements as given by the proper motions. The object of research is the proper motions of all stars of a magnitude brighter than 6.0 as taken from the correlation tables collected by CHARLIER from the *Preliminary General Catalogue* of Boss. Characteristics to the fourth order are given and some fundamental stellar constants are discussed.



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Introduction.

Ever since the day when it was announced that the peculiar motions of the stars are not distributed in a hap-hazard way to give preference to no certain direction, a distribution that had before been regarded as nearly an axiom, many astronomers have been working at the problems involved in defining the nature of this distribution and its bearings on our knowledge of the stellar universe. It has then been mainly two different hypotheses that have governed the ideas; the hypothesis of two star-streams proposed by KAPTEYN and the ellipsoidal hypothesis of SCHWARTZSCHILD. As long as it was only a matter of choosing a function of interpolation the two-stream hypothesis answered its purpose best as it involves one more parameter than the rival hypothesis. But when it comes to the bearings on the nature of the stellar system, the ellipsoidal hypothesis is to be preferred, it being by far the most simple of the two, and more fertile for generalisation. Besides the two-stream hypothesis supposes a dual character of the Milky Way, an assumption which it is repugnant to make without necessity.

When we take the material for the investigation from the proper motions, the question of the distribution in space of the stars is met with. Clearly here is the ground for more assumptions and hypotheses. To escape this question most authors on the subject have refrained from regarding the magnitudes of the motion contenting themselves with making a statistics of the *position angles*. However, one circumstance has hereby not been duly accentuated. That is, that the characteristics of the distribution obtained from the statistics of position angles do not necessarily refer to the *linear motions* of the stars. Really they are as much a property of the *apparent motion* or any other system of stellar motion and positions in space feasible to produce the observed proper motions. The only knowledge obtained is that if the assumptions regarding the qualities of distribution of the *linear motions* are true, then the parameters have the values found. Nothing in the nature of proving the truth of the assumptions is furnished.

In order to give a description as good as possible of the distribution of the linear peculiar motions of the stars we must regard also the amounts of the motions.

The assumptions regarding the density function and the luminosity curve of the stars which are then necessary need not be of any high standard of perfection. It is enough if they are fit as functions of interpolation. Fortunately the modern results of stellar statistics give quite enough for this purpose.

The first to attack the problem of reducing the attributes of the *apparent* cross-motions to those of the *linear* motions was CHARLIER¹. For the functions of density and luminosity CHARLIER made use of forms found in the most modern works of authors such as KAPTEYN, SEELIGER, SCHWARTZSCHILD and himself. He assumes the logarithm of the distances of the stars and the absolute magnitudes to be distributed as normal Gaussian frequency curves. Then it is shown that the characteristics (moments) of the *linear* velocities may be expressed in terms of the characteristics of the *apparent* velocities by means of only two constants. One of those constants is the mean parallax and it may be determined from the mean parallactic motion by comparing with the value of that motion found from the radial velocities. The chief difficulty arises when fixing the value of the other constant, which depends on the variation of the mean parallax for different apparent magnitudes. Having adopted a value of the constant, CHARLIER investigated the Boss' proper motions assuming as surface of equal frequency an ellipsoid of revolution.

It is now the aim of this work to seek for a more general characterisation of the distribution. The only assumption made here is that the frequency function is of the general statistical A-type, an assumption of ample elasticity. The characteristics are then computed to the fourth order. As the value of the fundamental constant adopted by CHARLIER is probably too low, and as it is difficult to fix any other definitive value otherwise than from the attributes of the motions themselves, I used a method by which I was able to leave the constants numerically indetermined, *thus obtaining the final results explicitly expressed in terms of the constants*. It then occurred to me that the fundamental constant might be obtained by comparing the results with those obtained from the radial velocities, of which an investigation has been simultaneously made by my colleague K. A. W. GYLLENBERG. As matters have turned out I have come to a modified opinion of the advisability of directly comparing the attributes of the cross-motions and the radial motions, but certain very interesting conclusions may still be drawn. However, other means of determining the constant with a high degree of probability offered themselves. It appeared that nearly all the higher coefficients change sign when varying the constant within a rather narrow range. As those coefficients are *coefficients of disturbancy* of the frequency function from its normal form this fact is of high interest, and makes it probable that the constant has a value within this range.

As before said the results depend only on the values of two statistical constants. The one is the mean parallax and is denoted by ϑ_1 , and the other is

¹ CHARLIER, Studies in Stellar statistics II. Medd. från Lunds Observatorium ser. II Nr 9.

called q' . As a working hypothesis I assumed as has done CHARLIER, that those parameters have the same value for the whole heavens. This assumption is necessarily very provisional. It is for instance exceedingly probable that the mean parallax varies with galactic latitude, though as we here only use stars brighter than the magnitude 6.0 the variation cannot be very great. However, I have been able, when studying the results, *to let the working hypothesis drop, having procured a method to correct the final results for any systematical variation of the constants*. The amount of variation of the parameter q' in different parts of the sky is very difficult to obtain, evidently the time is not yet ripe for such a determination. Besides it is in the present case of only slight importance. The variation of the mean parallax, however, I have tried to determine by comparing the mean parallaxic motion of the several regions in the sky with the mean parallaxic motion of the whole heavens. The correlation coefficients and the higher characteristics being abstract numbers (of dimension zero) they are practically independent of the variation. Only the axes of the correlation ellipsoid (mean internal motion) are affected and duly corrected.

CHAPTER I.

The frequencyfunction of the *A*-type for three variables.

1. According to the theory of mathematical statistics as developed principally by BRUNS and CHARLIER the frequency-function of a unitary statistical population with three variates x, y, z generally has the form

$$(1) \quad F(x, y, z) = \varphi(x, y, z) + \sum_{ijk} B_{ijk} \frac{\partial^{i+j+k} \varphi(x, y, z)}{\partial x^i \partial y^j \partial z^k}.$$

If x, y, z are the deviations from the means we have

$$\begin{aligned} \varphi(x, y, z) &= H e^{-1/2 f} \\ f &= A x^2 + B y^2 + C z^2 + 2 D y z + 2 E x z + 2 F x y \end{aligned}$$

and the quantities A, B, C, D, E, F may be determined in such a way that $i + j + k \geq 3$.

The function (1) is called the correlation function of the *A*-type, and the coefficient H is so chosen that:

$$(2) \quad \iiint_{-\infty}^{\infty} dx dy dz \varphi(x, y, z) = 1.$$

If $\nu_{\alpha\beta\gamma}$ are the *moments* about the mean of the function $F(x, y, z)$ we have

$$(3) \quad \iiint_{-\infty}^{\infty} dx dy dz x^\alpha y^\beta z^\gamma F(x, y, z) = \nu_{\alpha\beta\gamma}.$$

Now it may be shown that

$$(4) \quad \iiint_{-\infty}^{\infty} dx dy dz x^\alpha y^\beta z^\gamma \frac{\partial^{i+j+k} \varphi(x, y, z)}{\partial x^i \partial y^j \partial z^k} = (-1)^{i+j+k} \frac{\underline{\alpha} \quad \underline{\beta} \quad \underline{\gamma}}{\underline{\alpha-i} \quad \underline{\beta-j} \quad \underline{\gamma-k}} \lambda_{\alpha-i, \beta-j, \gamma-k}$$

when simultaneously $\alpha \geq i, \beta \geq j, \gamma \geq k$. Here we denote with $\lambda_{\alpha\beta\gamma}$ the moments of the function $\varphi(x, y, z)$. Whenever either $\alpha < i$ or $\beta < j$ or $\gamma < k$ the integral becomes equal to zero. As $\lambda_{\alpha\beta\gamma} = 0$ when $\alpha + \beta + \gamma$ is an odd number we see that the integral also disappears whenever $(\alpha - i) + (\beta - j) + (\gamma - k)$ is an odd number.

Hence we have for $\alpha + \beta + \gamma = 3$ or 4

$$(4^*) \quad \nu_{\alpha\beta\gamma} = \lambda_{\alpha\beta\gamma} + (-1)^{\alpha+\beta+\gamma} B_{\alpha\beta\gamma} \begin{vmatrix} \alpha \\ \beta \\ \gamma \end{vmatrix}$$

2. Before proceeding further we first write the exponent f in the form*:

$$(5) \quad f = Ax^2 + By^2 + Cz^2 + D_1yz + D_2zy + E_1zx + E_2xz + F_1xy + F_2yx$$

$$\text{where} \quad D_1 = D_2 = D \quad E_1 = E_2 = E \quad F_1 = F_2 = F$$

and define the determinants

$$(6) \quad \Delta = \begin{vmatrix} A & F_1 & E_1 \\ F_2 & B & D_1 \\ E_2 & D_2 & C \end{vmatrix}; \quad M = \begin{vmatrix} \lambda_{200} & \lambda_{110} & \lambda_{101} \\ \lambda_{110} & \lambda_{010} & \lambda_{011} \\ \lambda_{101} & \lambda_{011} & \lambda_{002} \end{vmatrix}$$

Making a linear orthogonal transformation of the variables thus

$$(7) \quad \begin{aligned} x &= a_{11}\xi + a_{12}\eta + a_{13}\zeta \\ y &= a_{21}\xi + a_{22}\eta + a_{23}\zeta \\ z &= a_{31}\xi + a_{32}\eta + a_{33}\zeta, \end{aligned}$$

we may chose the coefficients a_{ij} so that

$$f = s_1\xi^2 + s_2\eta^2 + s_3\zeta^2,$$

and the coefficients s are the roots of the equation

$$(8) \quad \begin{vmatrix} A-s & F_1 & E_1 \\ F_2 & B-s & D_1 \\ E_2 & D_2 & C-s \end{vmatrix} = 0.$$

Consequently we find

$$\Delta = s_1 s_2 s_3,$$

and as now the variables may be separated we find from the condition (2)

$$(9) \quad H = \frac{(s_1 s_2 s_3)^{1/2}}{(2\pi)^{3/2}} = \frac{\Delta^{1/2}}{(2\pi)^{3/2}}.$$

3. To find the parameters of the functions $F(x, y, z)$ expressed through the moments we first remark that for $\alpha + \beta + \gamma = 2$ we have $\lambda_{\alpha\beta\gamma} = \nu_{\alpha\beta\gamma}$ and consequently

$$M \equiv \begin{vmatrix} \nu_{200} & \nu_{110} & \nu_{101} \\ \nu_{110} & \nu_{020} & \nu_{011} \\ \nu_{101} & \nu_{011} & \nu_{002} \end{vmatrix}.$$

Now we write the equation (2) in the form

$$(2^*) \quad \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} dx dy dz e^{-1/2(Ax^2 + By^2 + Cz^2 + D_1yz + D_2zy + E_1xz + E_2xz + F_1xy + F_2yx)} = \frac{1}{\Delta^{1/2}}.$$

* For a part of the developments of this and the following paragraph compare GREINER: Zeitschrift für Mathematik und Physik Bd 57 P. 227, ff.

Differentiating this equation with regard to one of the parameters in the exponent and multiplying with $\Delta^{1/2}$ we obtain the moments. We then find, for instance, differentiating with regard to A ,

$$-^{1/2} \frac{\Delta^{1/2}}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} dx dy dz x^2 e^{1/2 f} = -^{1/2} \frac{1}{\Delta} \frac{\partial \Delta}{\partial A}$$

or

$$\lambda_{200} = \nu_{200} = \frac{\partial \Delta}{\partial A} \frac{1}{\Delta}.$$

In a similar way we find the other moments of the second order

$$(10) \quad \begin{aligned} \lambda_{200} = \nu_{200} &= \frac{\partial \Delta}{\partial A} \frac{1}{\Delta}; & \lambda_{110} = \nu_{110} &= \frac{\partial \Delta}{\partial F_1} \frac{1}{\Delta} \\ \lambda_{020} = \nu_{020} &= \frac{\partial \Delta}{\partial B} \frac{1}{\Delta}; & \lambda_{101} = \nu_{101} &= \frac{\partial \Delta}{\partial E_1} \frac{1}{\Delta} \\ \lambda_{002} = \nu_{002} &= \frac{\partial \Delta}{\partial C} \frac{1}{\Delta}; & \lambda_{011} = \nu_{011} &= \frac{\partial \Delta}{\partial D_1} \frac{1}{\Delta}. \end{aligned}$$

Using well known theorems it may now be shown that

$$(10^*) \quad M = \frac{1}{\Delta}$$

and that we have

$$(11) \quad \begin{aligned} A &= \frac{\partial M}{\partial \nu_{200}} \frac{1}{M}; & D &= \left(\frac{\partial M}{\partial \nu_{011}} \right) \frac{1}{M} \\ B &= \frac{\partial M}{\partial \nu_{020}} \frac{1}{M}; & E &= \left(\frac{\partial M}{\partial \nu_{101}} \right) \frac{1}{M} \\ C &= \frac{\partial M}{\partial \nu_{002}} \frac{1}{M}; & F &= \left(\frac{\partial M}{\partial \nu_{110}} \right) \frac{1}{M}, \end{aligned}$$

where the brackets denote that the differentiation is to be performed only for one of the places where the moments occur in the determinant M .

Finally we now have

$$(12) \quad \Pi = \frac{1}{M^{1/2} (2\pi)^{3/2}},$$

and as a check to the computations

$$(13) \quad A \nu_{200} + B \nu_{020} + C \nu_{002} + 2D \nu_{011} + 2E \nu_{101} + 2F \nu_{110} = 3.$$

Through the formulas (11) and (12) all the parameters of the generating function φ are expressed in terms of the moments.

4. To obtain the higher coefficients B_{ijk} expressed in a similar way we use the formula (4*). Then we first have to determine the moments $\lambda_{\alpha\beta\gamma}$ expressed in terms of the moments $\nu_{\alpha\beta\gamma} = \lambda_{\alpha\beta\gamma} (\alpha + \beta + \gamma = 2)$.

Here two different cases may be separated

1:0 when $\alpha + \beta + \gamma$ is an odd number.

Then clearly

$$(14) \quad \lambda_{\alpha\beta\gamma} = 0.$$

2:0 when $\alpha + \beta + \gamma$ is an even number.

Then we proceed as follows, taking for instance the case of the moment λ_{211} :

We differentiate the equation (2*) first according to A , then according to D , and obtain

$$^{1/4} \frac{1}{(2\pi)^{3/2}} \iiint_{-\infty}^{\infty} dx dy dz x^2 y z e^{-1/2 f} = \frac{1}{2} \cdot \frac{3}{2} \frac{1}{\Delta^{5/2}} \frac{\partial \Delta}{\partial A} \cdot \frac{\partial \Delta}{\partial D_1} - \frac{1}{2} \frac{1}{\Delta^{3/2}} \frac{\partial^2 \Delta}{\partial A \partial D_1},$$

or multiplying with $\Delta^{1/2}$

$$^{1/4} \lambda_{211} = ^{3/4} \lambda_{200} \lambda_{011} + ^{1/2} \frac{1}{\Delta} D_2.$$

Now as

$$\frac{1}{\Delta} = M \text{ and } D_2 = \left(\frac{\partial M}{\partial v_{011}} \right) \frac{1}{M},$$

we find

$$\frac{1}{\Delta} D_2 = \left(\frac{\partial M}{\partial \lambda_{011}} \right) = \lambda_{110} \lambda_{101} - \lambda_{011} \lambda_{200},$$

and consequently

$$\lambda_{211} = 3\lambda_{200} \lambda_{011} + 2\lambda_{110} \lambda_{101} - 2\lambda_{011} \lambda_{200} = v_{200} v_{011} + 2v_{110} v_{101}.$$

Similarly dealing with the other moments we find:

(15)

$$\begin{aligned} \lambda_{400} &= 3v_{200}^2 & \lambda_{103} &= 3v_{002} v_{101} & \lambda_{031} &= 3v_{020} v_{011} \\ \lambda_{310} &= 3v_{200} v_{110} & \lambda_{202} &= 2v_{101}^2 + v_{200} v_{002} & \lambda_{022} &= 2v_{011}^2 + v_{020} v_{002} \\ \lambda_{220} &= 2v_{110}^2 + v_{200} v_{020} & \lambda_{301} &= 3v_{200} v_{101} & \lambda_{013} &= 3v_{002} v_{011} \\ \lambda_{130} &= 3v_{020} v_{110} & \lambda_{004} &= 3v_{002}^2 \\ \lambda_{040} &= 3v_{020}^2 \\ \lambda_{211} &= 2v_{101} v_{110} + v_{200} v_{011} \\ \lambda_{121} &= 2v_{011} v_{110} + v_{020} v_{101} \\ \lambda_{112} &= 2v_{101} v_{011} + v_{002} v_{110} \end{aligned}$$

According to equation (4*) now

$$(16) \quad \begin{aligned} \underline{[3]} B_{210} &= -v_{300} & \underline{[2]} B_{201} &= -v_{201} \\ \underline{[2]} B_{210} &= -v_{210} & \underline{[2]} B_{102} &= -v_{102} \\ \underline{[2]} B_{120} &= -v_{120} & \underline{[3]} B_{003} &= -v_{003} \\ \underline{[3]} B_{030} &= -v_{030} \\ \underline{[2]} B_{021} &= -v_{021} & B_{111} &= -v_{111} \\ \underline{[2]} B_{012} &= -v_{012} \end{aligned}$$

and

$$\begin{aligned}
 & \begin{array}{l} \underline{[4]} \\ \underline{[3]} \\ \underline{[2]} \end{array} \begin{array}{l} B_{400} = v_{400} - 3v_{200}^2 \\ B_{310} = v_{310} - 3v_{200}v_{110} \\ B_{220} = v_{220} - 2v_{110}^2 - v_{200}v_{020} \end{array} \\
 & \begin{array}{l} \underline{[3]} \\ \underline{[4]} \\ \underline{[3]} \end{array} \begin{array}{l} B_{130} = v_{130} - 3v_{020}v_{110} \\ B_{040} = v_{040} - 3v_{020}^2 \\ B_{301} = v_{301} - 3v_{200}v_{101} \end{array} \\
 & \begin{array}{l} \underline{[2]} \\ \underline{[3]} \\ \underline{[4]} \end{array} \begin{array}{l} B_{202} = v_{202} - 2v_{101}^2 - v_{200}v_{002} \\ B_{103} = v_{103} - 3v_{002}v_{101} \\ B_{004} = v_{004} - 3v_{002}^2 \end{array} \\
 & \begin{array}{l} \underline{[3]} \\ \underline{[2]} \\ \underline{[3]} \end{array} \begin{array}{l} B_{013} = v_{013} - 3v_{011}v_{002} \\ B_{022} = v_{022} - 2v_{011}^2 - v_{020}v_{002} \\ B_{031} = v_{031} - 3v_{011}v_{020} \end{array} \\
 & \begin{array}{l} \underline{[2]} \\ \underline{[2]} \\ \underline{[2]} \end{array} \begin{array}{l} B_{211} = v_{211} - 2v_{101}v_{110} - v_{200}v_{011} \\ B_{121} = v_{121} - 2v_{110}v_{011} - v_{020}v_{101} \\ B_{112} = v_{112} - 2v_{101}v_{011} - v_{002}v_{110} \end{array}
 \end{aligned}
 \tag{17}$$

by which the expressions of the characteristics to the fourth order in terms of the moments are completed.

5. We here introduce the following notations

$$\frac{\partial^{i+j+k} \varphi(x, y, z)}{\partial x^i \partial y^j \partial z^k} = R_{ijk} \varphi(x, y, z),$$

where R_{ijk} is seen to be a polynom of the degree $i + j + k$ in x, y, z , and now may write

$$F(x, y, z) = \varphi(x, y, z) (1 + \sum B_{ijk} R_{ijk}).$$

Putting

$$\begin{aligned}
 v_{200} &= \sigma_x^2; & v_{020} &= \sigma_y^2; & v_{002} &= \sigma_z^2, \\
 \frac{v_{110}}{\sigma_x \sigma_y} &= r_{xy}; & \frac{v_{101}}{\sigma_x \sigma_z} &= r_{xz}; & \frac{v_{011}}{\sigma_y \sigma_z} &= r_{yz}, \\
 \frac{B_{ijk}}{\sigma_x^i \sigma_y^j \sigma_z^k} &= \beta_{ijk},
 \end{aligned}
 \tag{20}$$

σ_x, σ_y and σ_z are the dispersions of x, y and z ; r_{xy}, r_{xz}, r_{yz} are the correlation coefficients; further, for $i + j + k = 3$, β_{ijk} are the *coefficients of skewness* and for $i + j + k = 4$, β_{ijk} are called the *coefficients of excess*. They all constitute the characteristics to the fourth order of the frequency-function. To discriminate them from the parameters $A, B, C, D, E, F, B_{ijk}$, which also are called characteristics, we will always refer to them as the β -characteristics.

Regarding x, y, z and $F(x, y, z)$ as coordinates of a 4-dimensional system we introduce the dispersions $\sigma_x \sigma_y \sigma_z$ as units of x, y, z and define the following *normal coordinates*.

$$X = \frac{x}{\sigma_x}; \quad Y = \frac{y}{\sigma_y}; \quad Z = \frac{z}{\sigma_z}; \quad \sqrt{M} \cdot F(x, y, z) = \mathfrak{F}.$$

Putting further

$$R_{ijk} = \sigma_x^i \sigma_y^j \sigma_z^k R'_{ijk}$$

we can demonstrate that

$$(22) \quad \mathcal{F} = \varphi'(X, Y, Z) (1 + \sum \beta_{ijk} R'_{ijk}).$$

Here

$$\varphi' = \frac{1}{\sqrt{2\pi}^3} e^{-1/2 f'}$$

$$(22^*) \quad f' = \frac{1}{S} \left(X^2 \frac{\partial S}{\partial r_{xx}} + Y^2 \frac{\partial S}{\partial r_{yy}} + Z^2 \frac{\partial S}{\partial r_{zz}} + 2YZ \frac{\partial S}{\partial r_{yz}} + 2XZ \frac{\partial S}{\partial r_{xz}} + XY \frac{\partial S}{\partial r_{xy}} \right),$$

if by S we denote the determiniant

$$S = \begin{vmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{vmatrix},$$

and it is understood that $r_{xx} = r_{yy} = r_{zz} = 1$. Furthermore, it may be shown that the polynoms R'_{ijk} as parameters only contain the correlation-coefficients r_{xy} , r_{xz} , r_{yz} . Then, if as system of coordinates we choose a system where the correlation coefficients vanish, which may always be done by the linear transformation (7) and the cubic (8), *the functions φ' and R'_{ijk} are independent of any parameter whatsoever and consequently may once for all be tabulated.* As then the variables may be separated it will be seen that

$$R'_{ijk} = R'_{i00} \cdot R'_{0j0} \cdot R'_{00k},$$

and if

$$\varphi_0(X) = \frac{1}{\sqrt{2\pi}} e^{-1/2 X^2},$$

$$\varphi_0^{(i)}(X) = \varphi_0(X) R'_{i00}$$

we obtain

$$(23) \quad \mathcal{F}_0 = \varphi_0(X) \cdot \varphi_0(Y) \cdot \varphi_0(Z) + \sum_{i+j+k \geq 3} \beta_{ijk} \varphi_0^{(i)}(X) \varphi_0^{(j)}(Y) \varphi_0^{(k)}(Z).$$

The functions

$\Phi_1 = \sqrt{2\pi} \varphi_0(X)$, $\Phi_{i+1} = \sqrt{2\pi} \varphi_0^{(i)}(x)$, are tabulated by BRUNS* for all $i > 5$, and, save for $i=1$ and $i=2$ $i=5$, by CHARLIER**.

Thus the frequency of any »point» X, Y, Z may be easily calculated.

Without tables of the functions Φ_i the computations may be performed, remembering that in the system considered

$$(24) \quad \begin{aligned} R'_{100} &= -X \\ R'_{200} &= X^2 - 1 \\ R'_{300} &= 3X - X^3 \\ R'_{400} &= 3 - 6X^2 + X^4, \end{aligned}$$

* BRUNS: Wahrscheinlichkeitsrechnung und Kollektivmasslehre. Verl. B. G. Teubner 1906.

** CHARLIER: Reaserches into the theory of probability. Medd. från Lunds Obs. Ser. II N. 5.

and

$$(25) \quad \mathcal{F}_0 = \varphi_0(X) \cdot \varphi_0(Y) \cdot \varphi_0(Z) \left[1 + \sum_{i+j+k \geq 3} \beta_{ijk} R'_{i00} \cdot R'_{0j0} R'_{00k} \right].$$

Having now in a condensed form worked out the theory of frequency-functions of the *A*-type for three variables, I remark that the mode of demonstration is built upon the nice workings of GREINER, and that in all other respects the line of progress is an extension of the elaborations of CHARLIER regarding one and two variables.

6. Before leaving the more general discussion of the frequency-functions of the *A*-type I will mention a few theorems.

The equations

$$\frac{\partial \mathcal{F}}{\partial X} = 0; \quad \frac{\partial \mathcal{F}}{\partial Y} = 0; \quad \frac{\partial \mathcal{F}}{\partial Z} = 0$$

give the coordinates of the »mode».

Referred to the system of coordinates where the variables are uncorrelated, and supposing the coefficients β_{ijk} to be so small that $(\sum \beta_{ijk} R_{ijk})^2$ may be neglected, we shall find for the coordinates of the »mode»

$$(26) \quad \begin{aligned} X_1 &= 3\beta_{300} + \beta_{120} + \beta_{102} \\ Y_1 &= 3\beta_{030} + \beta_{210} + \beta_{012} \\ Z_1 &= 3\beta_{003} + \beta_{201} + \beta_{021}. \end{aligned}$$

The quantities, irrespective of the system of coordinates,

$$S_x = 3\beta_{300} \quad S_y = 3\beta_{030} \quad S_z = 3\beta_{003}$$

are called the *skewness* respectively of *x*, *y* and *z*.

In the equation (25) we put $X=0$, $Y=0$, $Z=0$ and find by help of (24)

$$\mathcal{F}(0, 0, 0) = \varphi_0(0)^3 [1 + 3\beta_{400} + 3\beta_{040} + 3\beta_{004} + \beta_{220} + \beta_{202} + \beta_{022}].$$

Consequently the sum

$$(32^{**}) \quad E = 3\beta_{400} + 3\beta_{040} + 3\beta_{004} + \beta_{220} + \beta_{202} + \beta_{022}$$

gives the relative *excess* over the normal frequency of the points (x, y, z) near the mean. Taking the frequency curves of the coordinates *x*, *y* and *z* separately we have for their excess

$$(32^*) \quad E_x = 3\beta_{400} \quad E_y = 3\beta_{040} \quad E_z = 3\beta_{004}.$$

The equations (23), (24), (25), (26) and (32**) are only valid for the system of coordinates where the variates are uncorrelated.

Regard, for instance, only the variates x and y , letting z have any value. Their frequency function will be obtained by integrating the function $F(x, y, z)$ for all z between $+\infty$ and $-\infty$.

Calling the resultant frequency function $F_1(x, y)$ we have

$$(27) \quad F_1(x, y) = \varphi_1(x, y) + \sum B_{ij} \frac{\partial^{i+j} \varphi_1(x, y)}{\partial x^i \partial y^j} \quad i+j \geq 3$$

where

$$(28) \quad \begin{aligned} \varphi_1(x, y) &= H_1 e^{-1/2 f_1} \\ f_1 &= A_1 x^2 + B_1 y^2 + 2F_1 xy, \end{aligned}$$

and similarly as by three variables writing

$$\delta = \begin{vmatrix} A_1 & F_1 \\ F_1 & B_1 \end{vmatrix}; \quad m = \begin{vmatrix} \nu_{20} & \nu_{11} \\ \nu_{11} & \nu_{02} \end{vmatrix},$$

we have

$$(28^*) \quad H_1 = \frac{1}{\sqrt{m} 2\pi}; \quad m = \frac{1}{\delta},$$

$$(29) \quad \begin{aligned} \nu_{20} &= \frac{\partial \delta}{\partial A_1} \frac{1}{\delta}; & A_1 &= \frac{\partial m}{\partial \nu_{20}} \frac{1}{m}, \\ \nu_{11} &= \left(\frac{\partial \delta}{\partial F_1} \right) \frac{1}{\delta}; & F_1 &= \left(\frac{\partial m}{\partial \nu_{11}} \right) \frac{1}{m}, \\ \nu_{02} &= \frac{\partial \delta}{\partial B_1} \frac{1}{\delta}; & B_1 &= \frac{\partial m}{\partial \nu_{02}} \frac{1}{m}. \end{aligned}$$

Now the parameters of $F_1(x, y)$ are obtained expressed in the parameters of $F(x, y, z)$ through the equation

$$\int_{-\infty}^{\infty} F(x, y, z) dz = F_1(x, y),$$

or more conveniently, by observing that

$$(29^*) \quad \nu_{ij} = \nu_{ijo},$$

through the equations

$$(30) \quad \begin{aligned} \frac{\partial \delta}{\partial A_1} \frac{1}{\delta} &= \frac{\partial \Delta}{\partial A} \frac{1}{\Delta} \\ \left(\frac{\partial \delta}{\partial F_1} \right) \frac{1}{\delta} &= \left(\frac{\partial \Delta}{\partial F} \right) \frac{1}{\Delta} \\ \frac{\partial \delta}{\partial B_1} \frac{1}{\delta} &= \frac{\partial \Delta}{\partial B} \frac{1}{\Delta}, \end{aligned}$$

and

$$(30^*) \quad B_{ij} = B_{ijo}.$$

The result will be found as given by CHARLIER (Studies in Stellar Statistics II p. 83):

$$(31) \quad \begin{aligned} CA_1 &= AC - E^2 \\ CB_1 &= BC - D^2 \\ CF_1 &= FC - DE. \end{aligned}$$

As

$$m = \frac{\partial M}{\partial \nu_{002}},$$

we find from (11) that

$$(32) \quad C = \frac{m}{M} = \frac{\Delta}{\varepsilon},$$

equations that will be used in chapter IV.

CHAPTER II.

The frequency function of the linear motions.

7. We denote by U, V, W the components of the linear motion of a star along any three rectangular axes. Then we write the frequency function according to (1) in the following way

$$(33) \quad F(U, V, W) = \varphi(U, V, W) + \sum B_{ijk} \frac{\partial^{i+j+k} \varphi(U, V, W)}{\partial U^i \partial V^j \partial W^k},$$

where as before

$$\varphi = \frac{1}{\sqrt{M}(2\pi)^{3/2}} e^{-1/2 f}$$

$$(33^*) \quad f = AU^2 + BV^2 + CW^2 + 2D VW + 2E UW + 2F UV.$$

We here denote the moments by N_{ijk} thus having

$$N_{ijk} = \iiint_{-\infty}^{\infty} U^i V^j W^k F(U, V, W) dU dV dW$$

or as it will be in practice

$$N_{ijk} = \text{Mean of } (U^i V^j W^k) = M(U^i V^j W^k).$$

The characteristics $A, B, C, D, E, F, B_{ijk}$ are given in terms of the moments N_{ijk} by the equations (11) (16) (17) when the ν_{ijk} are changed against the N_{ijk} .

The equation

$$f = \text{constant}$$

is the equation of the correlation ellipsoid or as it here is called *the velocity ellipsoid*.

By the way, we will here remark, that the correlation ellipsoid is not identical with the ellipsoid of SCHWARTZSCHILD even when the latter is taken three axial, except in the case when the higher characteristics all vanish. Generally they are both approximations to the surfaces of equal frequency, but not the same approximations. This circumstance is intimately connected with the fact hinted at in the introduction

that the ellipsoid obtained by only employing the position angles, regardless of the closeness of approximation, belongs as well to the *apparent* as to the *linear* motions, while the correlation ellipsoid has analytically different form for the two kinds of motion.

8. In the following we will refer to four different kinds of systems of coordinates. We denote them with the numbers I, II, III, IV, and they are defined as follows:

I. A system having its W -axis directed toward a point of the northern hemisphere and its V -axis in direction of growing northern declination.

Referring to this system we use the notations

$$U, V, W, A, B, C, D, E, F, B_{ijk}, N_{ijk}.$$

II. The usual astronomical system of coordinates having its U -axis directed toward the vernal equinox and its W -axis toward the northpole of the equator.

The notations are here

$$U'', V'', W'', A'', B'', C'', D'', E'', F'', B''_{ijk}, N''_{ijk}.$$

III. A galactic system of coordinates having its W -axis directed toward the north galactic pole and its U -axis towards the point $\alpha = 270^\circ$ $\delta = -15^\circ$.

Here we use notations with the index (s).

IV. The system of *vertices*, or the system where the function f has the form

$$f = A' U'^2 + B' V'^2 + C' W'^2,$$

using here notations with index '.

It will be seen in the practical part that III and IV only slightly differ.

The direction cosines of the systems referred to the system II are given according to the following schemes

I—II	U'' V'' W''	III—II	U'' V'' W''	IV—II	U'' V'' W''
U	γ_{11} γ_{21} γ_{31}	$U^{(s)}$	α_{11} α_{21} α_{31}	U'	ε_{11} ε_{21} ε_{31}
V	γ_{12} γ_{22} γ_{32}	$V^{(s)}$	α_{12} α_{22} α_{32}	V'	ε_{12} ε_{22} ε_{32}
W	γ_{13} γ_{23} γ_{33}	$W^{(s)}$	α_{13} α_{23} α_{33}	W'	ε_{13} ε_{23} ε_{33}

9. It will now be our concern to express the characteristics of the motion as referred to one system of coordinates in terms of the characteristics as referred to another system of coordinates. This constitutes the problem of the rotation of a frequency function and involves the chief algebraical labour of this memoir.

We take the case of transforming the frequency function of the components of motion in system II into the frequency function of system I. The relative

number of motions having components within the intervals $U'' \pm \frac{1}{2} dU''$; $V'' \pm \frac{1}{2} dV''$; $W'' \pm \frac{1}{2} dW''$ are expressed through

$$F''(U'', V'', W'') dU'' dV'' dW'',$$

and the number having components within the intervals $U \pm \frac{1}{2} dU$; $V \pm \frac{1}{2} dV$; $W \pm \frac{1}{2} dW$ through

$$F(U, V, W) dU dV dW,$$

and we have

$$(34) \quad \begin{aligned} U'' &= \gamma_{11} U + \gamma_{12} V + \gamma_{13} W \\ V'' &= \gamma_{21} U + \gamma_{22} V + \gamma_{23} W \\ W'' &= \gamma_{31} U + \gamma_{32} V + \gamma_{33} W. \end{aligned}$$

As the *Jacobiana* of this transformation is equal to 1 we have

$$dU dV dW = dU'' dV'' dW''$$

and accordingly

$$F(U, V, W) = F''(U'', V'', W'')$$

when in F'' we substitute the expressions (34).

Writing now

$$(35) \quad F''(U'', V'', W'') = \varphi''(U'', V'', W'') + \sum_{ijk} B''_{ijk} \frac{\partial^{i+j+k} \varphi''}{\partial U''^i \partial V''^j \partial W''^k}$$

$$\varphi'' = \frac{\Delta''^{1/2}}{\sqrt{2\pi^3}} e^{-1/2 f''}$$

$$(35^*) \quad f'' = A'' U''^2 + B'' V''^2 + C'' W''^2 + 2D'' V'' W'' + 2E'' U'' W'' + 2F'' U'' V''$$

$F(U, V, W)$ will also take this form.

By the substitution (34) the determinant Δ'' is invariant; consequently

$$\Delta'' = \Delta.$$

Substituting (34) in (35*) we obtain*

$$(36) \quad \begin{aligned} A &= A'' \gamma_{11}^2 + B'' \gamma_{21}^2 + C'' \gamma_{31}^2 + 2D'' \gamma_{21} \gamma_{31} + 2E'' \gamma_{11} \gamma_{31} + 2F'' \gamma_{11} \gamma_{21} \\ B &= A'' \gamma_{12}^2 + B'' \gamma_{22}^2 + C'' \gamma_{32}^2 + 2D'' \gamma_{22} \gamma_{32} + 2E'' \gamma_{12} \gamma_{32} + 2F'' \gamma_{12} \gamma_{22} \\ C &= A'' \gamma_{13}^2 + B'' \gamma_{23}^2 + C'' \gamma_{33}^2 + 2D'' \gamma_{23} \gamma_{33} + 2E'' \gamma_{13} \gamma_{33} + 2F'' \gamma_{13} \gamma_{23} \\ D &= A'' \gamma_{12} \gamma_{13} + B'' \gamma_{22} \gamma_{23} + C'' \gamma_{32} \gamma_{33} + D''(\gamma_{22} \gamma_{33} + \gamma_{32} \gamma_{23}) + E''(\gamma_{32} \gamma_{13} + \gamma_{33} \gamma_{12}) + F''(\gamma_{12} \gamma_{23} + \gamma_{13} \gamma_{22}) \\ E &= A'' \gamma_{13} \gamma_{11} + B'' \gamma_{23} \gamma_{21} + C'' \gamma_{33} \gamma_{31} + D''(\gamma_{23} \gamma_{31} + \gamma_{33} \gamma_{21}) + E''(\gamma_{33} \gamma_{11} + \gamma_{31} \gamma_{13}) + F''(\gamma_{13} \gamma_{21} + \gamma_{11} \gamma_{23}) \\ F &= A'' \gamma_{11} \gamma_{12} + B'' \gamma_{21} \gamma_{22} + C'' \gamma_{31} \gamma_{32} + D''(\gamma_{21} \gamma_{32} + \gamma_{31} \gamma_{22}) + E''(\gamma_{31} \gamma_{12} + \gamma_{32} \gamma_{11}) + F''(\gamma_{11} \gamma_{22} + \gamma_{12} \gamma_{21}) \end{aligned}$$

To transform the characteristics B''_{ijk} we introduce for a moment the following symbols

* Compare CHARLIER l. c. p. 82.

$$(37) \quad \begin{aligned} \frac{\partial}{\partial U'} &= D_1'' & \frac{\partial}{\partial V''} &= D_2'' & \frac{\partial}{\partial W''} &= D_3'' \\ \frac{\partial}{\partial U} &= D_1 & \frac{\partial}{\partial V} &= D_2 & \frac{\partial}{\partial W} &= D_3. \end{aligned}$$

Then clearly

$$(38) \quad \begin{aligned} D_1'' &= \gamma_{11} D_1 + \gamma_{12} D_2 + \gamma_{13} D_3 \\ D_2'' &= \gamma_{21} D_1 + \gamma_{22} D_2 + \gamma_{23} D_3 \\ D_3'' &= \gamma_{31} D_1 + \gamma_{32} D_2 + \gamma_{33} D_3, \end{aligned}$$

and using the symbols

$$(39) \quad \frac{\partial^{i+j+k}}{\partial U^i \partial V^j \partial W^k} = D_1^i D_2^j D_3^k,$$

according to a well known theorem

$$(40) \quad D_1''^i D_2''^j D_3''^k = (\gamma_{11} D_1 + \gamma_{12} D_2 + \gamma_{13} D_3)^i (\gamma_{21} D_1 + \gamma_{22} D_2 + \gamma_{23} D_3)^j (\gamma_{31} D_1 + \gamma_{32} D_2 + \gamma_{33} D_3)^k,$$

where after the development of the right membrum, we introduce the derivatives by the equation (39).

Now we immediately find the equation

$$\varphi(U, V, W) + \sum B_{ijk} D_1^i D_2^j D_3^k = \varphi''(U'', V'', W'') + \sum B_{ijk}' D_1''^i D_2''^j D_3''^k$$

or

$$(41) \quad \begin{aligned} &\sum B_{ijk} D_1^i D_2^j D_3^k = \\ &= \sum B_{ijk}' (\gamma_{11} D_1 + \gamma_{12} D_2 + \gamma_{13} D_3)^i (\gamma_{21} D_1 + \gamma_{22} D_2 + \gamma_{23} D_3)^j (\gamma_{31} D_1 + \gamma_{32} D_2 + \gamma_{33} D_3)^k. \end{aligned}$$

Equating here the coefficients of $D_1^\alpha D_2^\beta D_3^\gamma$ in both membra the expression of $B_{\alpha\beta\gamma}$ in terms of B_{ijk}' ($i+j+k = \alpha+\beta+\gamma$) is obtained.

Introducing the symbolical expressions

$$(42) \quad B_{\alpha\beta\gamma}' = \xi_1^\alpha \xi_2^\beta \xi_3^\gamma (\alpha, \beta, \gamma),$$

where by (α, β, γ) we mean the coefficient of $a^\alpha b^\beta c^\gamma$ in the development of

$$(a + b + c)^{\alpha+\beta+\gamma},$$

we generally have

$$(43) \quad B_{ijk} = (i, j, k) (\gamma_{11} \xi_1 + \gamma_{21} \xi_2 + \gamma_{31} \xi_3)^i (\gamma_{12} \xi_1 + \gamma_{22} \xi_2 + \gamma_{32} \xi_3)^j (\gamma_{13} \xi_1 + \gamma_{23} \xi_2 + \gamma_{33} \xi_3)^k,$$

when after the development we again introduce the characteristics $B_{\alpha\beta\gamma}'$ by (42).

Equations (36) and (43) give in a condensed form the whole theory of finding the characteristics when the system of coordinates is affected by a rotation.

CHAPTER III.

The method of Charlier to express the moments of the linear cross-motions through the moments of the proper-motions.

10. Taking out one region of the sky small enough to be regarded as plane the stars in the region constitute a statistical population comparable as to their projected motions. Taking as system of coordinates a system having its x -axis directed in the plane of the equator towards growing right ascension and its y -axis towards growing declination, the components of the proper motions we denote by $u + x_0$ and $v + y_0$. The components of the *linear* motion being $U + X_0$, $V + Y_0$ we have the equations

$$(44) \quad \begin{aligned} r \cdot (u + x_0) &= U + X_0 \\ r \cdot (v + y_0) &= V + Y_0 \end{aligned}$$

if r is the distance of the star.

If x_0 and y_0 are the mean proper motions and X_0 , Y_0 the mean linear motions u , v and U , V are respectively the *apparent* and the *linear peculiar* motions.

By forming the moments we proceed in the following way:

Putting as moments about the origin

$$\begin{aligned} v'_{ij} &= M((u + x_0)^i (v + y_0)^j) \\ N'_{ij} &= M((U + X_0)^i (V + Y_0)^j), \end{aligned}$$

and as moments about the mean

$$\begin{aligned} v_{ij} &= M(u^i v^j) \\ N_{ij} &= M(U^i V^j), \end{aligned}$$

we derive the equations

$$(45) \quad \begin{aligned} v'_{20} &= v_{20} + x_0^2 \\ v'_{11} &= v_{11} + x_0 y_0 \\ v'_{02} &= v_{02} + y_0^2 \end{aligned}$$

$$\begin{aligned}
 (45^*) \quad \nu'_{30} &= \nu_{30} + 3\nu_{20} x_0 + x_0^3 \\
 \nu'_{21} &= \nu_{21} + \nu_{20} y_0 + 2\nu_{11} x_0 + x_0^2 y_0 \\
 \nu'_{12} &= \nu_{12} + \nu_{02} x_0 + 2\nu_{11} y_0 + x_0 y_0^2 \\
 \nu'_{03} &= \nu_{03} + 3\nu_{02} y_0 + y_0^3
 \end{aligned}$$

$$\begin{aligned}
 (45^{**}) \quad \nu'_{40} &= \nu_{40} + 4\nu_{30} x_0 + 6\nu_{20} x_0^2 + x_0^4 \\
 \nu'_{31} &= \nu_{31} + 3\nu_{21} x_0 + \nu_{30} y_0 + 3\nu_{11} x_0^2 + 3\nu_{20} x_0 y_0 + x_0^3 y_0 \\
 \nu'_{22} &= \nu_{22} + 2\nu_{21} y_0 + 2\nu_{12} x_0 + 4\nu_{11} x_0 y_0 + \nu_{02} x_0^2 + \nu_{20} y_0^2 + x_0^2 y_0^2 \\
 \nu'_{13} &= \nu_{13} + 3\nu_{12} y_0 + \nu_{03} x_0 + 3\nu_{11} y_0^2 + 3\nu_{02} x_0 y_0 + x_0 y_0^3 \\
 \nu'_{04} &= \nu_{04} + 4\nu_{03} y_0 + 6\nu_{02} y_0^2 + y_0^4
 \end{aligned}$$

and *mutatis mutandis* the same equations for N'_{ij} .

On the other side the moments about the mean may be expressed in the moments about the origin by similar equations.

11. From equations (44) we have

$$\begin{aligned}
 u + x_0 &= \frac{1}{r} (U + X_0) \\
 v + y_0 &= \frac{1}{r} (V + Y_0).
 \end{aligned}$$

Assuming the linear velocities of the stars to be independent of the distance we thus have

$$\nu'_{ij} = M \left(\left(\frac{1}{r} \right)^{i+j} \right) N'_{ij},$$

or putting

$$M \left(\frac{1}{r^s} \right) = \vartheta_s,$$

we write

$$(46) \quad \nu'_{ij} = \vartheta_{i+j} N'_{ij}.$$

NOW CHARLIER uses the notations

$a(m) dm$ = the number of stars having their *apparent* magnitudes between $m + \frac{1}{2} dm$ and $m - \frac{1}{2} dm$,

$\varphi_0(M) dM$ = the number of stars having their *absolute* magnitudes between $M + \frac{1}{2} dM$ and $M - \frac{1}{2} dM$,

and putting

$$\begin{aligned}
 r &= e^{-by} & b &= 0.2/\text{mod} \\
 y &= -5 \log r
 \end{aligned}$$

$\Delta_0(y) dy$ = the number of stars having y between $y + \frac{1}{2} dy$ and $y - \frac{1}{2} dy$.
Then evidently as

$$M = m + y$$

we have

$$(47) \quad a(m) = \int_{-\infty}^{\infty} dy \Delta_0(y) \varphi_0(m+y)$$

$$(47^*) \quad a(m) \vartheta_s(m) = \int_{-\infty}^{\infty} dy \Delta_0(y) \varphi_0(m+y) e^{s \cdot dy},$$

where $\vartheta_s(m)$ is the mean of $\frac{1}{r^s}$ for the stars having the magnitude m .

Regarding the density function $\Delta_0(y)$ and the luminosity curve $\varphi_0(M)$ CHARLIER and others have found that they can be expressed by

$$(48) \quad \varphi_0(M) = \frac{1}{\sigma_2 \sqrt{2\pi}} e^{-\frac{(M - m_2)^2}{2\sigma_2^2}}$$

$$(48^*) \quad \Delta_0(y) = \frac{1}{\sigma_1 \sqrt{2\pi}} e^{-\frac{(y - m_1)^2}{2\sigma_1^2}}$$

by which he also finds from (47) and (47*)

$$(49) \quad a(m) = \frac{1}{k \sqrt{2\pi}} e^{-\frac{(m - m_0)^2}{2k^2}},$$

$$k^2 = \sigma_1^2 + \sigma_2^2$$

$$m_0 = m_2 - m_1,$$

and

$$(50) \quad \vartheta_1(m) = K \cdot e^{-b\lambda_1 m}$$

where

$$k^2 \lambda_1 = \sigma_1^2.$$

Finally putting

$$(50^{**}) \quad q' = e^{-b^2 k^2 \lambda_1 (1 - \lambda_1)},$$

he finds

$$(50^*) \quad \vartheta_s(m) = \vartheta_1^s(m) \left(\frac{1}{q'} \right)^{1/2 (s^2 - s)}.$$

12. In the present investigation we do not use stars of a certain magnitude m , but *all stars brighter than* $m = 6.0$.

The question is now: can the equation (50*) be used even in this case? CHARLIER seems to have been of the opinion that, taking stars brighter than 6.0, equation (50*) is still valid.

By numerical calculations I have come to the same opinion except on one single point. I find that equation (50**) should be written

$$(51) \quad q' = \alpha_2 e^{-(b^2 k^2 - \alpha_1) \lambda_1 (1 - \lambda_1)}$$

where α_2 and α_1 do not depend on λ_1 (at least when the limiting magnitude is ≤ 6.0).

Evidently the thing is of no consequence for our purpose as we determine the constant q' from the motions themselves, but it is of eminent importance when we wish to find the constant λ_1 from the value found for q' . Indeed, this is one of the most actual *desiderata* of stellar astronomy.

Clearly for all stars brighter than the magnitude m

$$(52) \quad \vartheta_s \int_{-\infty}^m a(m) dm = \left(\frac{1}{q'}\right)^{1/2 (s^2 - s)} \int_{-\infty}^m a(m) \vartheta_1^s(m) dm.$$

The consequences of this equation I have only studied numerically, assuming that for the stars brighter than 6.0 we may put

$$(53) \quad a(m) = K_1 3^m = K_1 e^{x \cdot m} \quad x = 1.1$$

Using this form for $a(m)$ we must, however, fix a lower limit for the magnitudes. Unfortunately the result will depend considerably on the choice of this limit. But as the stars brighter than 3.0 are only in number about $1/40$ of the stars brighter than 6.0 they will only slightly affect the value of ϑ_s . Further the number of stars brighter than 3.0 is too small for the formula (50) for the mean parallax to be valid. Accordingly as a lower limit we fix the magnitude 3.0. Then

$$\vartheta_s \int_{3.0}^{6.0} e^{xm} dm = \left(\frac{1}{q'}\right)^{1/2 (s^2 - s)} K^s \int_{3.0}^{6.0} e^{(-sb\lambda_1 + x)m} dm,$$

which gives

$$\vartheta_s = \left(\frac{1}{q'}\right)^{1/2 (s^2 - s)} \cdot K^s \frac{x}{(x - sb\lambda_1)} \cdot \frac{e^{6(x - sb\lambda_1)} - e^{3(x - sb\lambda_1)}}{e^{6x} - e^{3x}},$$

or putting

$$\frac{sb\lambda_1}{x} = \varepsilon,$$

$$\vartheta_s = \left(\frac{1}{q'}\right)^{1/2 (s^2 - s)} \vartheta_1^s \cdot \Pi_s,$$

we have

$$\Pi_s = \frac{(1 - \varepsilon)^s}{1 - \varepsilon^s} \frac{1 - e^{-3x(1 - \varepsilon s)}}{(1 - e^{-3x(1 - \varepsilon)})^s} (1 - e^{-3x})^{s-1}$$

The following small table will show the result for $s = 2$

λ_1	0.5	0.6	0.7	0.8	0.9	1.0
Π_2	1.03	1.04	1.06	1.08	1.11	1.14
p	1.03	1.03	1.05	1.07	1.10	1.14

The second line contains the values of the function

$$p = 1.14 e^{-\alpha_1(1-\lambda_1)\lambda_1}$$

where

$$\alpha_1 = 0.429$$

When computing the values of Π_3 and Π_4 it will be found that approximatively

$$\Pi_s = \Pi_2^{1/2(s^2 - s)}$$

Consequently we take account of the correction by writing

$$q' = 0.88 e^{-(b^2 k^2 - 0.43) \lambda_1 (1 - \lambda_1)}$$

and as before

$$(50^{***}) \quad \vartheta_s = \vartheta_1^s \left(\frac{1}{q'} \right)^{1/2(s^2 - s)}$$

It is easily seen that q' can not exceed the value 0.88 or, taking into account the arbitrary character of the magnitude 3.0 as a lower limit, at least not much exceed that value. Evidently the value $q' = 0.88$ corresponds to the value $\lambda_1 = 1$. As we have

$$\lambda_1 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}$$

this requires that

$$\sigma_2 = 0,$$

or that the absolute magnitude is the same for all stars apparently brighter than 6.0. A value of λ_1 only slightly smaller than 1, however, does not necessarily require the dispersion of the absolute magnitudes — σ_2 — to be very small, it only requires that it is small compared with the dispersions σ_1 of the quantity $y = -5 \log r$.

For the dispersion of the apparent magnitudes CHARLIER has for 12 regions of the heavens found $k = 3$, having made use of counts of stars as far down as to the magnitude 13.89.

For the stars that have magnitudes smaller than 6.0 we have put

$$a(m) = K_1 e^{xm} \quad x = 1.1$$

The general formula being

$$a(m) = \frac{1}{k \sqrt{2\pi}} e^{-\frac{(m - m_0)^2}{2k^2}}$$

we see, as

$$K_1 = a(0),$$

that for magnitudes small compared to m_0 we must have

$$\frac{m_0}{k^2} = x.$$

For m_0 we insert the mean of the values found by CHARLIER or

$$m_0 = 18.3,$$

from which we see that the stars brighter than 6.0 will best fit into a normal frequency curve having the dispersion

$$k = 4.1.$$

Indeed, by more recent computations performed at the *Lund Observatory* taking account of the results arrived at by HENIE for the distributions of the stars to the eleventh magnitude, it has been found questionable if the dispersion k should not have a value more near to four than to three. As in the following we shall make use of a q' amounting to $3/4$ it will be of interest to see what should be its bearings on λ_1 and σ_2 .

By the equation for q' of page 25 we find for $k = 4$

$$\lambda_1 = \frac{\sigma_1^2}{k^2} = 0.94,$$

$$\sigma_1 = 3.9 \quad \sigma_2 = 1.0.$$

Taking $k = 3$ we find $\lambda_1 = 0.87$ which gives

$$\sigma_1 = 2.8 \quad \sigma_2 = 1.1,$$

by which it is seen that for $q' = 0.75$ the value adopted for k will not have much influence on σ_2 .

The unities are for

$r : 1$ Siriometer $= 10^6$, the mean distance of the sun, and for

M : the same unit as for m ,

so that M corresponds to the magnitude of a star when placed at the distance 1 Siriometer.

The parallax is

$$\pi = \frac{1}{r} 0''.206265,$$

from which it follows that

$$M(\pi) = \vartheta_1.0''.206265.$$

Of course, as y and M are logarithms, σ_1 and σ_2 are independent of the units chosen for the distance and the luminosity.

12. Now, to find the moments of the linear motion in the region considered, we have

$$(54) \quad \vartheta_1^s N'_{ij} = q^{r^{1/2}(s^2 - s)} v'_{ij}$$

and using the equations (45) (45*) (45**) for v'_{ij} and N'_{ij} we find

$$(55) \quad \begin{aligned} \vartheta_1^2 N_{20} &= q' \nu_{20} - (1 - q') x_0^2 \\ \vartheta_1^2 N_{11} &= q' \nu_{11} - (1 - q') x_0 y_0 \\ \vartheta_1^2 N_{02} &= q' \nu_{02} - (1 - q') y_0^2 \end{aligned}$$

$$(55^*) \quad \begin{aligned} \vartheta_1^3 N_{30} &= \nu_{30} q'^3 - 3\nu_{20} x_0 (1 - q'^2) q' + x_0^3 (2 - 2q' + q'^3) \\ \vartheta_1^3 N_{21} &= \nu_{21} q'^3 - (y_0 \nu_{20} + 2x_0 \nu_{11}) (1 - q'^2) q' + x_0^2 y_0 (2 - 3q' + q'^3) \\ \vartheta_1^3 N_{12} &= \nu_{12} q'^3 - (x_0 \nu_{02} + 2y_0 \nu_{11}) (1 - q'^2) q' + x_0 y_0^2 (2 - 3q' + q'^3) \\ \vartheta_1^3 N_{03} &= \nu_{03} q'^3 - 3\nu_{02} y_0 (1 - q'^2) q' + y_0^3 (2 - 3q' + q'^3) \end{aligned}$$

$$(55^{**}) \quad \begin{aligned} \vartheta_1^4 (N_{40} - 3N_{20}^2) &= \nu_{40} q'^6 - 4\nu_{30} x_0 (1 - q'^3) q'^3 + 6\nu_{20} x_0^2 (2 - q' - 2q'^2 + q'^5) q' - \\ &\quad - 3\nu_{20}^2 q'^2 - x_0^4 (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ \vartheta_1^4 (N_{31} - 3N_{20} N_{11}) &= \nu_{31} q'^6 - (3\nu_{21} x_0 + \nu_{30} y_0) (1 - q'^3) q'^3 + \\ &\quad + 3(\nu_{11} x_0^2 + \nu_{20} x_0 y_0) (2 - q' - 2q'^2 + q'^5) q' - 3\nu_{20} \nu_{11} q'^2 - \\ &\quad - x_0^3 y_0 (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ \vartheta_1^4 (N_{22} - N_{20} N_{02} - 2N_{11}^2) &= \nu_{22} q'^6 - 2(\nu_{12} x_0 + \nu_{21} y_0) (1 - q'^3) q'^3 + \\ &\quad + (4\nu_{11} x_0 y_0 + \nu_{20} y_0^2 + \nu_{02} x_0^2) (2 - q' - 2q'^2 + q'^5) q' - \\ &\quad - (2\nu_{11}^2 + \nu_{20} \nu_{02}) q'^2 - x_0^2 y_0^2 (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ \vartheta_1^4 (N_{13} - 3N_{02} N_{11}) &= \nu_{13} q'^6 - (3\nu_{12} y_0 + \nu_{03} x_0) (1 - q'^3) q'^3 + \\ &\quad + 3(\nu_{11} y_0^2 + \nu_{02} x_0 y_0) (2 - q' - 2q'^2 + q'^5) q' - 3\nu_{02} \nu_{11} q'^2 - \\ &\quad - x_0 y_0^3 (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ \vartheta_1^4 (N_{04} - 3N_{02}^2) &= \nu_{04} q'^6 - 4\nu_{03} y_0 (1 - q'^3) q'^3 + 6\nu_{02} y_0^2 (2 - q' - 2q'^2 + q'^5) q' - \\ &\quad - 3\nu_{02}^2 q'^2 - y_0^4 (6 - 12q' + 3q'^2 + 4q'^3 - q'^6). \end{aligned}$$

By those equations the moments of the linear motions for the stars within a small area of the sky, are expressed through the moments of the apparent motion.

CHAPTER IV.

The determination of the frequency distribution of the velocities in space from the proper motions.

13. Dividing the sky with CHARLIER in 48 »squares» of equal area symmetrically distributed with regard to the equator, and letting the direction cosines γ_{13} , γ_{23} , γ_{33} be the direction cosines of the centre of gravity of the square, we have as system of coordinates of each pair of diametrically situated squares the system denoted by I. Such a pair I will in the following refer to as simply a »square».

Having computed the moments of the *apparent* proper motions of a square we obtain by the equations (55) (55*) (55***) the moments of the linear cross motions expressed in q' and ϑ_1 .

Now the general line of progress will be the following:

By help of the equations (29) and the five first of the equations (17) we determine the characteristics of the motion as projected on the square. Then by equations (31) and (30*) we express those 12 characteristics in terms of the 31 characteristics of the motion in space referred to the system I of each square. Hereafter by aid of equations (36) and (43) we get the characteristics of motion as projected on the axes of system I expressed through the characteristics of the motion referred to the system II. Accordingly we obtain from each square 12 equations of condition between the 31 unknown characteristics of the motion in space referred to system II. Evidently the problem of finding these 31 characteristics is mathematically determinate if we have recourse to observations in three squares. For the characteristics of the second order — that is the ellipsoid — even two squares are enough. Now it will presently be shown that *all the equations are linear*, and naturally we then apply the method of least squares to the material from the whole heavens.

For the characteristics of the ellipsoid we thus compute normal equations for six unknowns from 72 equations of condition.

For the characteristics of the third order: 10 unknowns from 96 equations, and for the characteristics of the fourth order: 15 unknowns from 120 equations.

By the way, it may be remarked already here that on account of the symmetry of the squares the normal equations take very simple forms. Thus it will never be necessary to solve systems of normal equations with more than 6 unknowns and even that only in one single case.

14. We will first develop the method to find the characteristics of the second order in system II and to compute the vertices and the three axes of the velocity ellipsoid. The method was worked out by the author in January last year and a preliminary notice was published shortly afterwards ¹.

Simultaneously a method using only two of the three characteristics available in each square was published by GYLLENBERG in connection with his studies of the radial motions ².

Up to that time only a method assuming an ellipsoid of revolution had been worked out by CHARLIER. Indeed, CHARLIER makes some general remarks concerning the solution of the problem of the three-axial ellipsoid, but on the whole he seems to have regarded it as too complicate ³.

From the equations (36) CHARLIER (p. 83) has developed the formulæ

$$\begin{aligned}
 A_1 C &= AC - E^2 = (B''C'' - D''^2)\gamma_{12}^2 + (C''A'' - E''^2)\gamma_{22}^2 + (A''B'' - F''^2)\gamma_{32}^2 \\
 &\quad + 2(D''E'' - C''F'')\gamma_{12}\gamma_{22} + 2(E''F'' - A''D'')\gamma_{22}\gamma_{32} + 2(F''D'' - B''E'')\gamma_{32}\gamma_{12} \\
 B_1 C &= BC - D^2 = (B''C'' - D''^2)\gamma_{11}^2 + (C''A'' - E''^2)\gamma_{21}^2 + (A''B'' - F''^2)\gamma_{31}^2 \\
 &\quad + 2(D''E'' - C''F'')\gamma_{11}\gamma_{21} + 2(E''F'' - A''D'')\gamma_{21}\gamma_{31} + 2(F''D'' - B''E'')\gamma_{31}\gamma_{11} \\
 -CF_1 &= -(FC - DE) = (B''C'' - D''^2)\gamma_{11}\gamma_{12} + (C''A'' - E''^2)\gamma_{21}\gamma_{22} + (A''B'' - F''^2)\gamma_{31}\gamma_{32} \\
 &\quad + (D''E'' - C''F'')(\gamma_{11}\gamma_{22} + \gamma_{12}\gamma_{21}) \\
 &\quad + (E''F'' - A''D'')(\gamma_{21}\gamma_{32} + \gamma_{22}\gamma_{31}) \\
 &\quad + (F''D'' - B''E'')(\gamma_{31}\gamma_{12} + \gamma_{32}\gamma_{11}).
 \end{aligned}$$

Theoretically, if the quantities A_1 , B_1 , F_1 are known for two squares, and as C by equation (36) is a linear function of A'' , B'' , C'' , D'' , E'' , F'' it seems as if the solution of those last quantities would lead to equations of the twelfth degree.

The whole difficulty consequently lies in the factor C of the left membrum. But we already have derived a property of C that will help us from this difficulty.

We have written

$$\Delta = \begin{vmatrix} A & F & E \\ F & B & D \\ E & D & C \end{vmatrix} \qquad \delta_1 = \begin{vmatrix} A_1 & F_1 \\ F_1 & B_1 \end{vmatrix}$$

and in the § 6 we found the equation

$$(32) \qquad C = \frac{\Delta}{\delta_1}.$$

¹ S. D. WICKSELL: A general Method to determine the three axes of the velocity ellipsoid from the proper motions of the stars. Medd. f. Lunds Observ. N:o 60.

² K. A. W. GYLLENBERG: On the three axial distribution of the velocities of the stars. Medd. fr. Lunds Observatorium N:o 59.

³ CHARLIER: Stellar statistics II, chapter II.

Now we know that the determinant Δ is invariant when changing the direction of the axes of coordinates. Thus

$$C = \frac{\Delta''}{\delta_1}.$$

Further we have, as

$$m = \frac{1}{\delta_1},$$

$$\frac{A_1}{\delta_1} = N_{02}; \quad \frac{B_1}{\delta_1} = N_{20}; \quad \frac{F_1}{\delta_1} = N_{11},$$

and writing

$$x_1 = \frac{B''C'' - D''^2}{\Delta''} \vartheta_1^2; \quad y_1 = \frac{A''C'' - E''^2}{\Delta''} \vartheta_1^2; \quad z_1 = \frac{A''B'' - F''^2}{\Delta''} \vartheta_1^2;$$

$$p_1 = \frac{D'E'' - C''F''}{\Delta''} \vartheta_1^2; \quad q_1 = \frac{E''F'' - A''D''}{\Delta''} \vartheta_1^2; \quad r_1 = \frac{F''D'' - B''E''}{\Delta''} \vartheta_1^2,$$

we obtain for $x_1, y_1, z_1, p_1, q_1, r_1$ the linear equations

$$(57) \quad \begin{aligned} \vartheta_1^2 N_{02} &= x_1 \gamma_{12}^2 + y_1 \gamma_{22}^2 + z_1 \gamma_{32}^2 + 2p_1 \gamma_{12} \gamma_{22} + 2q_1 \gamma_{22} \gamma_{32} + 2r_1 \gamma_{12} \gamma_{32} \\ \vartheta_1^2 N_{20} &= x_1 \gamma_{11}^2 + y_1 \gamma_{21}^2 + z_1 \gamma_{31}^2 + 2p_1 \gamma_{11} \gamma_{21} + 2q_1 \gamma_{21} \gamma_{31} + 2r_1 \gamma_{11} \gamma_{31} \\ \vartheta_1^2 N_{11} &= x_1 \gamma_{11} \gamma_{12} + y_1 \gamma_{21} \gamma_{22} + z_1 \gamma_{31} \gamma_{32} + p_1 (\gamma_{11} \gamma_{22} + \gamma_{12} \gamma_{21}) + q_1 (\gamma_{21} \gamma_{32} + \gamma_{22} \gamma_{31}) \\ &\quad + r_1 (\gamma_{31} \gamma_{12} + \gamma_{32} \gamma_{11}). \end{aligned}$$

Inserting the expressions (55), and as $\gamma_{31} = 0$, we get

$$(58) \quad \begin{aligned} \nu_{02} q' - y_0^2 (1 - q') &= x_1 \alpha_1 + y_1 \alpha_2 + z_1 \alpha_3 + p_1 \alpha_4 + q_1 \alpha_5 + r_1 \alpha_6 \\ \nu_{20} q' - x_0^2 (1 - q') &= x_1 \beta_1 + y_1 \beta_2 + p_1 \beta_4 \\ \nu_{11} q' - x_0 y_0 (1 - q') &= x_1 \gamma_1 + y_1 \gamma_2 + p_1 \gamma_4 + q_1 \gamma_5 + r_1 \gamma_6, \end{aligned}$$

where

$$\begin{aligned} \alpha_1 &= \gamma_{12}^2 & \alpha_2 &= \gamma_{22}^2 & \alpha_3 &= \gamma_{32}^2 & \alpha_4 &= 2\gamma_{12} \gamma_{22} & \alpha_5 &= 2\gamma_{22} \gamma_{32} & \alpha_6 &= 2\gamma_{12} \gamma_{32} \\ \beta_1 &= \gamma_{11}^2 & \beta_2 &= \gamma_{21}^2 & & & \beta_4 &= 2\gamma_{11} \gamma_{21} & & & \\ \gamma_1 &= \gamma_{11} \gamma_{12} & \gamma_2 &= \gamma_{21} \gamma_{22} & & & \gamma_4 &= (\gamma_{11} \gamma_{22} + \gamma_{12} \gamma_{21}) & \gamma_5 &= \gamma_{21} \gamma_{32} & \gamma_6 &= \gamma_{11} \gamma_{32} \end{aligned}$$

On account of $\gamma_{31} = 0$ we have besides

$$\gamma_1 = -\gamma_2, \quad \gamma_5 = +\gamma_{13}, \quad \gamma_6 = -\gamma_{23}.$$

Letting q' remain undetermined we get normal equations having the left membrum of the form

$$a_1 q' - b_1 (1 - q').$$

Accordingly the quantities $x_1, y_1, z_1, p_1, q_1, r_1$ are expressed in the same way.

To more clearly point out the *genesis* of the equations (57) we remark that according to (10) we have

$$(59) \quad \begin{aligned} x_1 &= \vartheta_1^2 N_{200}'; & y_1 &= \vartheta_1^2 N_{020}'; & z_1 &= \vartheta_1^2 N_{002}' \\ p_1 &= \vartheta_1^2 N_{110}'; & q_1 &= \vartheta_1^2 N_{011}'; & r_1 &= \vartheta_1^2 N_{101}'. \end{aligned}$$

Further from (11)

$$(60) \quad \begin{aligned} A'' &= \vartheta_1^2 \frac{y_1 z_1 - q_1^2}{k_3} & B'' &= \vartheta_1^2 \frac{x_1 z_1 - r_1^2}{k_3} & C'' &= \vartheta_1^2 \frac{x_1 y_1 - p_1^2}{k_3} \\ D'' &= \vartheta_1^2 \frac{p_1 r_1}{k_3} - x_1 q_1 & E'' &= \vartheta_1^2 \frac{p_1 q_1}{k_3} - y_1 r_1 & F'' &= \vartheta_1^2 \frac{r_1 q_1}{k_3} - p_1 z_1, \end{aligned}$$

when

$$k_3 = \vartheta_1^6 M = \begin{vmatrix} x_1, & p_1, & r_1 \\ p_1, & y_1, & q_1 \\ r_1, & q_1, & z_1 \end{vmatrix} = x_1 y_1 z_1 - x_1 q_1^2 - y_1 r_1^2 - z_1 p_1^2 + 2p_1 q_1 r_1.$$

Having found $A'', B'', C'', D'', E'', F''$ we obtain the axes of the ellipsoid from the equation

$$\begin{vmatrix} A'' - \xi, & F'', & E'' \\ F'', & B'' - \xi, & D'' \\ E'', & D'', & C'' - \xi \end{vmatrix} = 0,$$

where, the roots being ξ_1, ξ_2, ξ_3 , we have

$$\xi_1 = A' \quad \xi_2 = B' \quad \xi_3 = C',$$

and the equation of the ellipsoid

$$A' U'^2 + B' V'^2 + C' W'^2 = 1.$$

However, to obtain the axes of the ellipsoid we need never make the computation (60). Indeed, on account of the formula (10), we have

$$\Delta'' = \frac{\vartheta_1^6}{k_3},$$

and it follows that the roots s_1, s_2, s_3 of the equation

$$(61) \quad \begin{vmatrix} x_1 - s, & p_1, & r_1 \\ p_1, & y_1 - s, & q_1 \\ r_1, & q_1, & z_1 - s \end{vmatrix} = 0$$

are equal to

$$s_1 = \frac{\vartheta_1^2}{A'} \quad s_2 = \frac{\vartheta_1^2}{B'} \quad s_3 = \frac{\vartheta_1^2}{C'},$$

or, calling the axes of the ellipsoid $\sigma_1, \sigma_2, \sigma_3$, we have

$$(61^{**}) \quad s_1 = \vartheta_1^2 \sigma_1^2 \quad s_2 = \vartheta_1^2 \sigma_2^2 \quad s_3 = \vartheta_1^2 \sigma_3^2.$$

Writing the cubic (61)

$$(61^*) \quad s^3 - k_1 s^2 + k_2 s - k_3 = 0$$

we have for the coefficients

$$(62) \quad \begin{aligned} k_1 &= x_1 + y_1 + z_1 \\ k_2 &= x_1 y_1 + x_1 z_1 + y_1 z_1 - p_1^2 - q_1^2 - r_1^2 \\ k_3 &= x_1 y_1 z_1 - x_1 q_1^2 - y_1 r_1^2 - z_1 p_1^2 + 2p_1 q_1 r_1. \end{aligned}$$

At last we write down the equations for the direction cosines of the vertices. In accordance with the theory of linear substitutions they are:

$$(63) \quad \begin{aligned} \text{Direction cos. of } \sigma_1 & \left\{ \begin{aligned} (x_1-s_1) \varepsilon_{11} + p_1 \varepsilon_{21} + r_1 \varepsilon_{31} &= 0 \\ p_1 \varepsilon_{11} + (y_1-s_1) \varepsilon_{21} + q_1 \varepsilon_{31} &= 0 \\ r_1 \varepsilon_{11} + q_1 \varepsilon_{21} + (z_1-s_1) \varepsilon_{31} &= 0 \end{aligned} \right. \\ \text{Direction cos. of } \sigma_2 & \left\{ \begin{aligned} (x_1-s_2) \varepsilon_{12} + p_1 \varepsilon_{22} + r_1 \varepsilon_{32} &= 0 \\ p_1 \varepsilon_{12} + (y_1-s_2) \varepsilon_{22} + q_1 \varepsilon_{32} &= 0 \\ r_1 \varepsilon_{12} + q_1 \varepsilon_{22} + (z_1-s_2) \varepsilon_{32} &= 0 \end{aligned} \right. \\ \text{Direction cos. of } \sigma_3 & \left\{ \begin{aligned} (x_1-s_3) \varepsilon_{13} + p_1 \varepsilon_{23} + r_1 \varepsilon_{33} &= 0 \\ p_1 \varepsilon_{13} + (y_1-s_3) \varepsilon_{23} + q_1 \varepsilon_{33} &= 0 \\ r_1 \varepsilon_{13} + q_1 \varepsilon_{23} + (z_1-s_3) \varepsilon_{33} &= 0. \end{aligned} \right. \end{aligned}$$

And for the declination and right ascension of the vertices we have

Vertex I	Vertex II	Vertex III
$\cos \delta_1 \cos \alpha_1 = \varepsilon_{11}$	$\cos \delta_2 \cos \alpha_2 = \varepsilon_{12}$	$\cos \delta_3 \cos \alpha_3 = \varepsilon_{13}$
$\cos \delta_1 \sin \alpha_1 = \varepsilon_{21}$	$\cos \delta_2 \sin \alpha_2 = \varepsilon_{22}$	$\cos \delta_3 \sin \alpha_3 = \varepsilon_{23}$
$\sin \delta_1 = \varepsilon_{31}$	$\sin \delta_2 = \varepsilon_{32}$	$\sin \delta_3 = \varepsilon_{33}$

15. Using the equations (41) or (43) we obtain as equations of condition for the characteristics of the third order, from each square a system of the following form:

(64)

$$\begin{aligned} B_{30} &= a_1 B''_{300} + a_2 B''_{030} + a_3 B''_{003} + a_4 B''_{210} + a_5 B''_{201} + a_6 B''_{120} + a_7 B''_{021} + a_8 B''_{102} + a_9 B''_{012} + a_{10} B''_{111} \\ B_{21} &= b_1 B''_{300} + b_2 B''_{030} + b_3 B''_{003} + b_4 B''_{210} + b_5 B''_{201} + b_6 B''_{120} + b_7 B''_{021} + b_8 B''_{102} + b_9 B''_{012} + b_{10} B''_{111} \\ B_{12} &= c_1 B''_{300} + c_2 B''_{030} + c_3 B''_{003} + c_4 B''_{210} + c_5 B''_{201} + c_6 B''_{120} + c_7 B''_{021} + c_8 B''_{102} + c_9 B''_{012} + c_{10} B''_{111} \\ B_{03} &= d_1 B''_{300} + d_2 B''_{030} + d_3 B''_{003} + d_4 B''_{210} + d_5 B''_{201} + d_6 B''_{120} + d_7 B''_{021} + d_8 B''_{102} + d_9 B''_{012} + d_{10} B''_{111} \end{aligned}$$

where the coefficients are given by: (observe $\gamma_{31} = 0$)

$a_1 = \gamma_{11}^3$	$b_1 = 3\gamma_{11}^3 \gamma_{12}$	$c_1 = 3\gamma_{11}^3 \gamma_{12}$	$d_1 = \gamma_{12}^3$
$a_2 = \gamma_{21}^3$	$b_2 = 3\gamma_{21}^3 \gamma_{22}$	$c_2 = 3\gamma_{21}^3 \gamma_{22}$	$d_2 = \gamma_{22}^3$
$a_3 = \gamma_{31}^3 = 0$	$b_3 = 3\gamma_{31}^3 \gamma_{32} = 0$	$c_3 = 3\gamma_{31}^3 \gamma_{32} = 0$	$d_3 = \gamma_{32}^3$
$a_4 = \gamma_{11}^2 \gamma_{21}$	$b_4 = 2\gamma_{11}^2 \gamma_{12} \gamma_{21} + \gamma_{11}^2 \gamma_{22}$	$c_4 = 2\gamma_{11}^2 \gamma_{12} \gamma_{22} + \gamma_{12}^2 \gamma_{21}$	$d_4 = \gamma_{12}^2 \gamma_{22}$
$a_5 = \gamma_{11}^2 \gamma_{31} = 0$	$b_5 = 2\gamma_{11}^2 \gamma_{12} \gamma_{31} + \gamma_{11}^2 \gamma_{32}$	$c_5 = 2\gamma_{11}^2 \gamma_{12} \gamma_{32} + \gamma_{12}^2 \gamma_{31}$	$d_5 = \gamma_{12}^2 \gamma_{32}$
$a_6 = \gamma_{11} \gamma_{21}^2$	$b_6 = 2\gamma_{11} \gamma_{21}^2 \gamma_{22} + \gamma_{12} \gamma_{21}^2$	$c_6 = 2\gamma_{21}^2 \gamma_{22} \gamma_{12} + \gamma_{11} \gamma_{22}^2$	$d_6 = \gamma_{12} \gamma_{22}^2$
$a_7 = \gamma_{21}^2 \gamma_{31} = 0$	$b_7 = 2\gamma_{21}^2 \gamma_{22} \gamma_{31} + \gamma_{21}^2 \gamma_{32}$	$c_7 = 2\gamma_{21}^2 \gamma_{22} \gamma_{32} + \gamma_{22}^2 \gamma_{31}$	$d_7 = \gamma_{22}^2 \gamma_{32}$
$a_8 = \gamma_{11} \gamma_{31}^2 = 0$	$b_8 = 2\gamma_{11} \gamma_{31}^2 \gamma_{32} + \gamma_{12} \gamma_{31}^2 = 0$	$c_8 = 2\gamma_{12} \gamma_{31}^2 \gamma_{32} + \gamma_{11} \gamma_{32}^2$	$d_8 = \gamma_{12} \gamma_{32}^2$
$a_9 = \gamma_{21} \gamma_{31}^2 = 0$	$b_9 = 2\gamma_{21} \gamma_{31}^2 \gamma_{32} + \gamma_{22} \gamma_{31}^2 = 0$	$c_9 = 2\gamma_{22} \gamma_{31}^2 \gamma_{32} + \gamma_{21} \gamma_{32}^2$	$d_9 = \gamma_{22} \gamma_{32}^2$
$a_{10} = \gamma_{11} \gamma_{21} \gamma_{31} = 0$	$b_{10} = \gamma_{11} \gamma_{21} \gamma_{32} + \gamma_{11} \gamma_{22} \gamma_{31} + \gamma_{12} \gamma_{21} \gamma_{31}$	$c_{10} = \gamma_{11} \gamma_{22} \gamma_{32} + \gamma_{12} \gamma_{21} \gamma_{32} + \gamma_{12} \gamma_{22} \gamma_{31}$	$d_{10} = \gamma_{12} \gamma_{22} \gamma_{32}$

By equations (55*) (30*) and the four first of equations (16) we have for the observed membra the forms

$$(65) \quad \begin{aligned} \vartheta_1^3 B_{30} &= a_{11} q'^3 + a_{12} (1 - q'^2) q' + a_{13} (2 - 3q' + q'^3) \\ \vartheta_1^3 B_{21} &= b_{11} q'^3 + b_{12} (1 - q'^2) q' + b_{13} (2 - 3q' + q'^3) \\ \vartheta_1^3 B_{12} &= c_{11} q'^3 + c_{12} (1 - q'^2) q' + c_{13} (2 - 3q' + q'^3) \\ \vartheta_1^3 B_{03} &= d_{11} q'^3 + d_{12} (1 - q'^2) q' + d_{13} (2 - 3q' + q'^3) \end{aligned}$$

where the coefficients are expressed in the moments of *apparent* motion as follows:

$$\begin{aligned} a_{11} &= -\nu_{30} : 6 & b_{11} &= -\nu_{12} : 2 \\ a_{12} &= +\nu_{20} x_0 : 2 & b_{12} &= + (y_0 \nu_{20} + 2x_0 \nu_{11}) : 2 \\ a_{13} &= -x_0^3 : 6 & b_{13} &= -x_0^2 y_0 : 2 \\ \\ c_{11} &= -\nu_{12} : 2 & d_{11} &= -\nu_{03} : 6 \\ c_{12} &= (x_0 \nu_{02} + 2y_0 \nu_{11}) : 2 & d_{12} &= \nu_{02} y_0 : 2 \\ c_{13} &= -x_0 y_0^2 : 2 & d_{13} &= -y_0^3 : 6 \end{aligned}$$

Applying the method of least squares to (64) and letting ϑ_1 and q' remain undetermined we finally arrive at the forms for the unknowns

$$(65^*) \quad \vartheta_1^3 B_{ijk} = H_{ijk}^{(1)} q'^3 + H_{ijk}^{(2)} (1 - q'^2) q' + H_{ijk}^{(3)} (1 - 3q' + q'^3) \\ i + j + k = 3.$$

16. The corresponding equations of condition for the characteristics of the fourth order may be written for each square:

(66)

$$\begin{aligned} B_{40} &= a'_1 B''_{400} + a'_2 B''_{040} + a'_3 B''_{004} + a'_4 B''_{130} + a'_5 B''_{103} + a'_6 B''_{310} + a'_7 B''_{301} + \\ &+ a'_8 B''_{013} + a'_9 B''_{031} + a'_{10} B''_{220} + a'_{11} B''_{202} + a'_{12} B''_{022} + a'_{13} B''_{211} + \\ &+ a'_{14} B''_{121} + a'_{15} B''_{112}, \\ B_{04} &= b'_1 B''_{400} + b'_2 B''_{040} + b'_3 B''_{004} + b'_4 B''_{130} + b'_5 B''_{103} + b'_6 B''_{310} + b'_7 B''_{301} + \\ &+ b'_8 B''_{013} + b'_9 B''_{031} + b'_{10} B''_{220} + b'_{11} B''_{202} + b'_{12} B''_{022} + b'_{13} B''_{211} + \\ &+ b'_{14} B''_{121} + b'_{15} B''_{112}, \\ B_{31} &= c'_1 B''_{400} + c'_2 B''_{040} + c'_3 B''_{004} + c'_4 B''_{130} + c'_5 B''_{103} + c'_6 B''_{310} + c'_7 B''_{301} + \\ &+ c'_8 B''_{013} + c'_9 B''_{031} + c'_{10} B''_{220} + c'_{11} B''_{202} + c'_{12} B''_{022} + c'_{13} B''_{211} + \\ &+ c'_{14} B''_{121} + c'_{15} B''_{112}, \\ B_{13} &= d'_1 B''_{400} + d'_2 B''_{040} + d'_3 B''_{004} + d'_4 B''_{130} + d'_5 B''_{103} + d'_6 B''_{310} + d'_7 B''_{301} + \\ &+ d'_8 B''_{013} + d'_9 B''_{031} + d'_{10} B''_{220} + d'_{11} B''_{202} + d'_{12} B''_{022} + d'_{13} B''_{211} + \\ &+ d'_{14} B''_{121} + d'_{15} B''_{112}, \\ B_{22} &= e'_1 B''_{400} + e'_2 B''_{040} + e'_3 B''_{004} + e'_4 B''_{130} + e'_5 B''_{103} + e'_6 B''_{310} + e'_7 B''_{301} + \\ &+ e'_8 B''_{013} + e'_9 B''_{031} + e'_{10} B''_{220} + e'_{11} B''_{202} + e'_{12} B''_{022} + e'_{13} B''_{211} + \\ &+ e'_{14} B''_{121} + e'_{15} B''_{112}, \end{aligned}$$

and the coefficients are expressed in the direction cosines γ_{ij} as follows ($\gamma_{31} = 0$):

$$\begin{aligned}
a'_1 &= \gamma_{11}^4 & b'_1 &= \gamma_{12}^4 & c'_1 &= 4\gamma_{11}^3 \gamma_{12} & d'_1 &= 4\gamma_{11} \gamma_{12}^3 & e'_1 &= 6\gamma_{11}^2 \gamma_{12}^2 \\
a'_2 &= \gamma_{21}^4 & b'_2 &= \gamma_{22}^4 & c'_2 &= 4\gamma_{21}^3 \gamma_{22} & d'_2 &= 4\gamma_{21} \gamma_{22}^3 & e'_2 &= 6\gamma_{21}^2 \gamma_{22}^2 \\
a'_3 &= 0 & b'_3 &= \gamma_{32}^4 & c'_3 &= 0 & d'_3 &= 0 & e'_3 &= 0 \\
a'_4 &= \gamma_{21}^3 \gamma_{11} & b'_4 &= \gamma_{22}^3 \gamma_{12} & c'_4 &= \gamma_{12} \gamma_{21}^3 + 3\gamma_{11} \gamma_{22} \gamma_{21}^2 & d'_4 &= \gamma_{11} \gamma_{22}^3 + 3\gamma_{12} \gamma_{21} \gamma_{22}^2 & e'_4 &= 3\gamma_{21} \gamma_{22}^2 \gamma_{11} + 3\gamma_{12} \gamma_{21}^2 \gamma_{22} \\
a'_5 &= 0 & b'_5 &= \gamma_{32}^3 \gamma_{12} & c'_5 &= 0 & d'_5 &= \gamma_{11} \gamma_{32}^3 & e'_5 &= 0 \\
a'_6 &= \gamma_{11}^3 \gamma_{21} & b'_6 &= \gamma_{12}^3 \gamma_{22} & c'_6 &= \gamma_{11} \gamma_{22}^3 + 3\gamma_{21} \gamma_{12} \gamma_{11}^2 & d'_6 &= \gamma_{21} \gamma_{12}^3 + 3\gamma_{22} \gamma_{11} \gamma_{12}^2 & e'_6 &= 3\gamma_{11} \gamma_{12}^2 \gamma_{21} + 3\gamma_{22} \gamma_{11}^2 \gamma_{12} \\
a'_7 &= 0 & b'_7 &= \gamma_{12}^3 \gamma_{32} & c'_7 &= \gamma_{32} \gamma_{11}^3 & d'_7 &= 3\gamma_{32} \gamma_{11} \gamma_{12}^2 & e'_7 &= 3\gamma_{32} \gamma_{11}^2 \gamma_{12} \\
a'_8 &= 0 & b'_8 &= \gamma_{32}^3 \gamma_{22} & c'_8 &= 0 & d'_8 &= \gamma_{21} \gamma_{32}^3 & e'_8 &= 0 \\
a'_9 &= 0 & b'_9 &= \gamma_{22}^3 \gamma_{32} & c'_9 &= \gamma_{32} \gamma_{21}^3 & d'_9 &= 3\gamma_{32} \gamma_{21} \gamma_{22}^2 & e'_9 &= 3\gamma_{32} \gamma_{21}^2 \gamma_{22} \\
a'_{10} &= \gamma_{11}^2 \gamma_{21}^2 & b'_{10} &= \gamma_{12}^2 \gamma_{22}^2 & c'_{10} &= 2\gamma_{11} \gamma_{12} \gamma_{21}^2 + 2\gamma_{11}^2 \gamma_{21} \gamma_{22} & d'_{10} &= 2\gamma_{11} \gamma_{12} \gamma_{22}^2 + 2\gamma_{12}^2 \gamma_{21} \gamma_{22} & e'_{10} &= \gamma_{12}^2 \gamma_{21}^2 + \gamma_{11}^2 \gamma_{22}^2 + 4\gamma_{11} \gamma_{12} \gamma_{21} \gamma_{22} \\
a'_{11} &= 0 & b'_{11} &= \gamma_{12}^2 \gamma_{32}^2 & c'_{11} &= 0 & d'_{11} &= 2\gamma_{11} \gamma_{12} \gamma_{32}^2 & e'_{11} &= \gamma_{11}^2 \gamma_{32}^2 \\
a'_{12} &= 0 & b'_{12} &= \gamma_{22}^2 \gamma_{32}^2 & c'_{12} &= 0 & d'_{12} &= 2\gamma_{21} \gamma_{22} \gamma_{32}^2 & e'_{12} &= \gamma_{21}^2 \gamma_{32}^2 \\
a'_{13} &= 0 & b'_{13} &= \gamma_{12}^2 \gamma_{22} \gamma_{32} & c'_{13} &= \gamma_{11} \gamma_{21} \gamma_{32}^2 & d'_{13} &= \gamma_{12}^2 \gamma_{21} \gamma_{32} + 2\gamma_{11} \gamma_{12} \gamma_{22} \gamma_{32} & e'_{13} &= \gamma_{11}^2 \gamma_{22} \gamma_{32} + 2\gamma_{11} \gamma_{12} \gamma_{21} \gamma_{32} \\
a'_{14} &= 0 & b'_{14} &= \gamma_{12} \gamma_{22}^2 \gamma_{32} & c'_{14} &= \gamma_{11} \gamma_{21}^2 \gamma_{32} & d'_{14} &= \gamma_{22}^2 \gamma_{11} \gamma_{32} + 2\gamma_{22} \gamma_{21} \gamma_{12} \gamma_{32} & e'_{14} &= \gamma_{21}^2 \gamma_{12} \gamma_{32} + 2\gamma_{21} \gamma_{22} \gamma_{11} \gamma_{32} \\
a'_{15} &= 0 & b'_{15} &= \gamma_{12} \gamma_{22} \gamma_{32}^2 & c'_{15} &= 0 & d'_{15} &= \gamma_{32}^3 \gamma_{11} \gamma_{21} + \gamma_{32}^2 \gamma_{12} \gamma_{21} & e'_{15} &= \gamma_{32}^2 \gamma_{11} \gamma_{21}
\end{aligned}$$

The coefficients have, however, not been computed from those formulæ, but from the following that may easily be verified

$$\begin{aligned}
a'_1 &= \beta_1^2 & b'_1 &= \alpha_1^2 & c'_1 &= 4\beta_1 \gamma_1 & d'_1 &= 4\alpha_1 \gamma_1 & e'_1 &= 6\alpha_1 \beta_1 \\
a'_2 &= \beta_2^2 & b'_2 &= \alpha_2^2 & c'_2 &= 4\beta_2 \gamma_2 & d'_2 &= 4\alpha_2 \gamma_2 & e'_2 &= 6\alpha_2 \beta_2 \\
a'_3 &= 0 & b'_3 &= \alpha_3^2 & c'_3 &= 0 & d'_3 &= 0 & e'_3 &= 0 \\
a'_4 &= 1/2 \beta_1 \beta_4 & b'_4 &= 1/2 \alpha_2 \alpha_4 & c'_4 &= \beta_2 \gamma_4 + \beta_4 \gamma_2 & d'_4 &= \alpha_2 \gamma_4 + \alpha_4 \gamma_2 & e'_4 &= 8/9 (\alpha_2 \beta_4 + \alpha_4 \beta_2) \\
a'_5 &= 0 & b'_5 &= 1/2 \alpha_3 \alpha_6 & c'_5 &= 0 & d'_5 &= \alpha_3 \gamma_6 & e'_5 &= 0 \\
a'_6 &= 1/2 \beta_1 \beta_4 & b'_6 &= 1/2 \alpha_1 \alpha_4 & c'_6 &= \beta_1 \gamma_4 + \beta_4 \gamma_1 & d'_6 &= \alpha_1 \gamma_4 + \alpha_4 \gamma_1 & e'_6 &= 3/2 (\alpha_1 \beta_4 + \alpha_4 \beta_1) \\
a'_7 &= 0 & b'_7 &= 1/2 \alpha_1 \alpha_6 & c'_7 &= \beta_1 \gamma_6 & d'_7 &= \alpha_1 \gamma_6 + \alpha_6 \gamma_1 = 3\alpha_1 \gamma_6 & e'_7 &= 8/2 \beta_1 \alpha_6 \\
a'_8 &= 0 & b'_8 &= 1/2 \alpha_3 \alpha_5 & c'_8 &= 0 & d'_8 &= \alpha_3 \gamma_5 & e'_8 &= 0 \\
a'_9 &= 0 & b'_9 &= 1/2 \alpha_2 \alpha_5 & c'_9 &= \beta_2 \gamma_5 & d'_9 &= \alpha_2 \gamma_5 + \alpha_5 \gamma_2 = 3\alpha_2 \gamma_5 & e'_9 &= 8/2 \beta_2 \alpha_5 \\
a'_{10} &= \beta_1 \beta_2 & b'_{10} &= \alpha_1 \alpha_2 & c'_{10} &= 2\gamma_1 \beta_2 + 2\beta_1 \gamma_2 & d'_{10} &= 2\gamma_1 \alpha_2 + 2\alpha_1 \gamma_2 & e'_{10} &= \alpha_1 \beta_2 + \alpha_2 \beta_1 - 4\alpha_1 \beta_1 \\
a'_{11} &= 0 & b'_{11} &= \alpha_1 \alpha_3 & c'_{11} &= 0 & d'_{11} &= 2\alpha_3 \gamma_1 & e'_{11} &= \alpha_3 \beta_1 \\
a'_{12} &= 0 & b'_{12} &= \alpha_2 \alpha_3 & c'_{12} &= 0 & d'_{12} &= 2\alpha_3 \gamma_2 & e'_{12} &= \alpha_3 \beta_2 \\
a'_{13} &= 0 & b'_{13} &= 1/2 \alpha_1 \alpha_5 & c'_{13} &= \beta_1 \gamma_5 & d'_{13} &= \alpha_1 \gamma_5 + \alpha_5 \gamma_1 & e'_{13} &= 1/2 (\alpha_5 \beta_1 + \beta_5 \alpha_6) \\
a'_{14} &= 0 & b'_{14} &= 1/2 \alpha_2 \alpha_6 & c'_{14} &= \beta_2 \gamma_6 & d'_{14} &= \alpha_2 \gamma_6 + \alpha_6 \gamma_2 & e'_{14} &= 1/2 (\beta_2 \alpha_6 + \beta_4 \alpha_5) \\
a'_{15} &= 0 & b'_{15} &= 1/2 \alpha_3 \alpha_4 & c'_{15} &= 0 & d'_{15} &= \alpha_3 \gamma_4 & e'_{15} &= 1/2 \alpha_3 \beta_4
\end{aligned}$$

Similarly as in the case of the characteristics of the third order we now write the left membra thus

(67)

$$\begin{aligned} B_{40} &= a'_{16} q'^6 + a'_{17} (1 - q'^3) q'^3 + a'_{18} (2 - q' - 2q'^2 + q'^5) q' + a'_{19} q'^2 + a'_{20} (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ B_{04} &= b'_{16} q'^6 + b'_{17} (1 - q'^3) q'^3 + b'_{18} (2 - q' - 2q'^2 + q'^5) q' + b'_{19} q'^2 + b'_{20} (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ B_{31} &= c'_{16} q'^6 + c'_{17} (1 - q'^3) q'^3 + c'_{18} (2 - q' - 2q'^2 + q'^5) q' + c'_{19} q'^2 + c'_{20} (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ B_{13} &= d'_{16} q'^6 + d'_{17} (1 - q'^3) q'^3 + d'_{18} (2 - q' - 2q'^2 + q'^5) q' + d'_{19} q'^2 + d'_{20} (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \\ B_{22} &= e'_{16} q'^6 + e'_{17} (1 - q'^3) q'^3 + e'_{18} (2 - q' - 2q'^2 + q'^5) q' + e'_{19} q'^2 + e'_{20} (6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \end{aligned}$$

and we find from (55**), (30*) and (17)

$$\begin{aligned} a'_{16} &= + \nu_{40} : 24 & b'_{16} &= + \nu_{04} : 24 & c'_{16} &= + \nu_{31} : 6 \\ a'_{17} &= - \nu_{30} x_0 : 6 & b'_{17} &= - \nu_{03} y_0 : 6 & c'_{17} &= - (3\nu_{21} x_0 + \nu_{30} y_0) : 6 \\ a'_{18} &= + \nu_{20} x_0^2 : 4 & b'_{18} &= + \nu_{02} y_0^2 : 4 & c'_{18} &= + (\nu_{11} x_0^2 + \nu_{20} x_0 y_0) : 2 \\ a'_{19} &= - \nu_{20}^2 : 8 & b'_{19} &= - \nu_{02}^2 : 8 & c'_{19} &= - \nu_{20} \nu_{11} : 2 \\ a'_{20} &= - x_0^4 : 24 & b'_{20} &= - y_0^4 : 24 & c'_{20} &= - x_0^3 y_0 : 6 \end{aligned}$$

$$\begin{aligned} d'_{16} &= + \nu_{13} : 6 & e'_{16} &= + \nu_{22} : 4 \\ d'_{17} &= - (3\nu_{12} y_0 + \nu_{03} x_0) : 6 & e'_{17} &= - (\nu_{12} x_0 + \nu_{21} y_0) : 2 \\ d'_{18} &= + (\nu_{11} y_0^2 + \nu_{02} x_0 y_0) : 2 & e'_{18} &= + (4\nu_{11} x_0 y_0 + \nu_{20} y_0^2 + \nu_{02} x_0^2) : 4 \\ d'_{19} &= - \nu_{02} \nu_{11} : 2 & e'_{19} &= - (2\nu_{11}^2 + \nu_{20} \nu_{02}) : 4 \\ d'_{20} &= - x_0 y_0^3 : 6 & e'_{20} &= - x_0^2 y_0^2 : 4. \end{aligned}$$

Having computed and solved the normal equations we will have as solutions the numerical values of the H_{ijk} in the following expressions:

$$(67^*) \quad \vartheta_1^4 B''_{ijk} = H_{ijk}^{(1)} q'^6 + H_{ijk}^{(2)} (1 - q'^3) q'^3 + H_{ijk}^{(3)} (2 - q' - 2q'^2 + q'^5) q' + H_{ijk}^{(4)} q'^2 + H_{ijk}^{(5)} (6 - 12q' + 3q'^2 + 4q'^3 - q'^6).$$

17. Now it would seem to be very laborious to undertake to compute and solve normal equations with fifteen unknowns especially as the unknowns come forth as algebraical polynoms of q' . However, a multitude of circumstances are at hand to facilitate the work in an astounding way. I will here mention a few that depend on the symmetrical distribution of the squares with regard to the equator.

When computing the normal equations we have to form sums of products of the coefficients, for instance $\Sigma a'_i a'_j$, over all the squares. Now it may be shown that in *all cases except five, the sums of products are zero or expressible in the sums of the quadrats of some of the other coefficients*. Indeed, it may easily be verified that

$$\begin{aligned} \Sigma \alpha_1 \alpha_2 &= 1/4 \Sigma \alpha_4^2, & \Sigma \alpha_1 \alpha_3 &= 1/4 \Sigma \alpha_6^2, & \Sigma \alpha_2 \alpha_3 &= 1/4 \Sigma \alpha_5^2 \\ \Sigma \beta_1 \beta_2 &= 1/4 \Sigma \beta_4^2, & \Sigma \gamma_1 \gamma_2 &= - \Sigma \gamma_1^2. \end{aligned}$$

All the other 23 sums of products are zero and here consequently only the sums of the quadrats need be computed.

Further we have:

$$\begin{aligned}
 \Sigma a_2 a_4 &= \Sigma a_6^2 = \Sigma a_4^2 \\
 \Sigma a_1 a_6 &= \Sigma a_6^2 = \Sigma a_4^2 \\
 \Sigma b_1 b_6 &= -\frac{1}{3} \Sigma b_1^3 = -\frac{1}{3} \Sigma b_2^3 \\
 \Sigma c_1 c_6 &= -\frac{1}{3} \Sigma c_1^2 = -\frac{1}{3} \Sigma c_2^2 \\
 \Sigma c_2 c_4 &= -\frac{1}{3} \Sigma c_1^2 \\
 \Sigma c_1 c_8 &= \frac{3}{4} \Sigma c_5^2 \\
 \Sigma c_6 c_8 + \Sigma c_4 c_9 &= \Sigma c_{10}^2 - \frac{1}{2} \Sigma c_5^2
 \end{aligned}
 \quad
 \begin{aligned}
 \Sigma b_2 b_4 &= -\frac{1}{3} \Sigma b_1^2 \\
 \Sigma b_5 b_7 &= \Sigma b_{10}^2 \\
 \Sigma c_2 c_9 &= \frac{3}{4} \Sigma c_7^2 = \frac{3}{4} \Sigma c_5^2 \\
 \Sigma c_5 c_7 &= -\Sigma c_5^2
 \end{aligned}$$

$$\begin{aligned}
 \Sigma d_2 d_4 &= \Sigma d_6^2 & \Sigma d_3 d_7 &= \Sigma d_9^2 & \Sigma d_4 d_9 &= \Sigma d_{10}^2 \\
 \Sigma d_1 d_6 &= \Sigma d_6^2 & \Sigma d_1 d_8 &= \Sigma d_5^2 & \Sigma d_5 d_7 &= \Sigma d_{10}^2 \\
 \Sigma d_3 d_5 &= \Sigma d_8^2 & \Sigma d_2 d_9 &= \Sigma d_7^2 & \Sigma d_6 d_8 &= \Sigma d_{10}^2
 \end{aligned}$$

and all the other 87 sums of products are zero; consequently here only the sum $\Sigma c_4 c_9$ need be computed directly, else only sums of quadrats are needed.

By the equations for the characteristics of the fourth order we finally get:

$$\begin{aligned}
 \Sigma a'_1 a'_2 &= \Sigma a'_{10}^2, & \Sigma a'_1 a'_{10} &= \Sigma a'_6^2, & \Sigma a'_2 a'_{10} &= \Sigma a'_4^2, & \Sigma a'_4 a'_6 &= \Sigma a'_{10}^2 \\
 \Sigma b'_4 b'_6 &= \Sigma b'_{10}^2, & \Sigma b'_4 b'_{15} &= \Sigma b'_{14}^2, & \Sigma b'_5 b'_7 &= \Sigma b'_{11}^2, & \Sigma b'_5 b'_{14} &= \Sigma b'_{15}^2 \\
 \Sigma b'_6 b'_{15} &= \Sigma b'_{13}^2, & \Sigma b'_7 b'_{14} &= \Sigma b'_{13}^2, & \Sigma b'_8 b'_9 &= \Sigma b'_{12}^2, & \Sigma b'_8 b'_{13} &= \Sigma b'_{15}^2 \\
 \Sigma b'_9 b'_{13} &= \Sigma b'_{14}^2, & \Sigma b'_{10} b'_{11} &= \Sigma b'_{13}^2, & \Sigma b'_{10} b'_{12} &= \Sigma b'_{14}^2, & \Sigma b'_{11} b'_{12} &= \Sigma b'_{15}^2 \\
 \Sigma b'_1 b'_2 &= \Sigma b'_{10}^2, & \Sigma b'_1 b'_3 &= \Sigma b'_{11}^2, & \Sigma b'_1 b'_{10} &= \Sigma b'_6^2, & \Sigma b'_1 b'_{11} &= \Sigma b'_7^2 \\
 \Sigma b'_1 b'_{12} &= \Sigma b'_{13}^2, & \Sigma b'_2 b'_3 &= \Sigma b'_{12}^2, & \Sigma b'_2 b'_{10} &= \Sigma b'_4^2, & \Sigma b'_2 b'_{11} &= \Sigma b'_{14}^2 \\
 \Sigma b'_2 b'_{12} &= \Sigma b'_9^2, & \Sigma b'_3 b'_{10} &= \Sigma b'_{15}^2, & \Sigma b'_3 b'_{11} &= \Sigma b'_5^2, & \Sigma b'_3 b'_{12} &= \Sigma b'_8^2
 \end{aligned}$$

$$\begin{aligned}
 \Sigma c'_1 c'_2 &= 2 \Sigma c'_{10}^2 - \frac{1}{2} \Sigma c'_1^2 - \frac{1}{2} \Sigma c'_2^2, & \Sigma c'_1 c'_{10} &= -\Sigma c'_{10}^2 - \frac{1}{4} \Sigma c'_1^2 + \frac{1}{4} \Sigma c'_2^2 \\
 \Sigma c'_4 c'_6 &= \frac{5}{4} \Sigma c'_{10}^2 - \frac{1}{8} \Sigma c'_1^2 - \frac{1}{8} \Sigma c'_2^2, & \Sigma c'_2 c'_{10} &= -\Sigma c'_{10}^2 + \frac{1}{4} \Sigma c'_1^2 - \frac{1}{4} \Sigma c'_2^2 \\
 \Sigma c'_7 c'_{14} &= \Sigma c'_{13}^2, & \Sigma c'_9 c'_{13} &= \Sigma c'_{14}^2
 \end{aligned}$$

$$\begin{aligned}
 \Sigma d'_1 d'_2 &= 2 \Sigma d'_{10}^2 - \frac{1}{2} \Sigma d'_1^2 - \frac{1}{2} \Sigma d'_2^2, & \Sigma d'_4 d'_6 &= \frac{5}{4} \Sigma d'_{10}^2 - \frac{1}{8} \Sigma d'_1^2 - \frac{1}{8} \Sigma d'_2^2 \\
 \Sigma d'_1 d'_{10} &= -\Sigma d'_{10}^2 - \frac{1}{4} \Sigma d'_1^2 + \frac{1}{4} \Sigma d'_2^2, & \Sigma d'_4 d'_{15} &= \Sigma d'_{14}^2 - \frac{1}{9} \Sigma d'_7^2 \\
 \Sigma d'_2 d'_{10} &= -\Sigma d'_{10}^2 + \frac{1}{4} \Sigma d'_1^2 - \frac{1}{4} \Sigma d'_2^2, & \Sigma d'_6 d'_{15} &= \Sigma d'_{13}^2 - \frac{1}{9} \Sigma d'_9^2 \\
 \Sigma d'_{10} d'_{11} &= \frac{4}{9} (\Sigma d'_9^2 - \Sigma d'_7^2), & \Sigma d'_{10} d'_{12} &= -\frac{4}{9} (\Sigma d'_9^2 - \Sigma d'_7^2) \\
 \Sigma d'_1 d'_{11} &= \frac{8}{9} \Sigma d'_7^2, & \Sigma d'_5 d'_7 &= \frac{3}{4} \Sigma d'_{11}^2, & \Sigma d'_1 d'_{12} &= -\frac{8}{9} \Sigma d'_7^2 \\
 \Sigma d'_2 d'_{11} &= -\frac{8}{9} \Sigma d'_9^2, & \Sigma d'_2 d'_{12} &= \frac{8}{9} \Sigma d'_9^2, & \Sigma d'_{11} d'_{12} &= -\Sigma d'_{11}^2
 \end{aligned}$$

$$\begin{aligned}
 \Sigma e'_1 e'_{10} &= \frac{2}{3} \Sigma e'_6^2 - \frac{1}{3} \Sigma e'_1^2, & \Sigma e'_4 e'_{14} &= \frac{1}{8} \Sigma e'_7^2 - \frac{1}{8} \Sigma e'_9^2 \\
 \Sigma e'_6 e'_{15} &= \frac{1}{3} \Sigma e'_9^2 - \frac{1}{3} \Sigma e'_7^2, & \Sigma e'_7 e'_{14} &= \frac{1}{3} \Sigma e'_9^2 - \frac{2}{3} \Sigma e'_7^2 \\
 \Sigma e'_9 e'_{13} &= \frac{1}{3} \Sigma e'_7^2 - \frac{2}{3} \Sigma e'_9^2, & \Sigma e'_2 e'_{10} &= \frac{2}{3} \Sigma c'_4^2 - \frac{1}{3} \Sigma e'_2^2 \\
 \Sigma e'_{10} e'_{10} &= \Sigma e'_{13}^2 - \frac{1}{3} \Sigma e'_9^2, & \Sigma e'_{10} e'_{12} &= \Sigma e'_{14}^2 - \frac{1}{3} \Sigma e'_7^2 \\
 \Sigma e'_1 e'_2 &= \Sigma e'_1^2, & \Sigma e'_4 e'_6 &= \Sigma e'_4^2, & \Sigma e'_1 e'_{11} &= \frac{2}{3} \Sigma e'_7^2, & \Sigma e'_1 e'_{12} &= \frac{2}{3} \Sigma e'_4^2 \\
 & & \Sigma e'_{11} e'_{12} &= \Sigma e'_{15}^2.
 \end{aligned}$$

Finally $\sum d'_7 d'_{14}$, $\sum d'_8 d'_{13}$, $\sum d'_5 d'_{14}$, $\sum d'_9 d'_{13}$ have to be computed directly. All the other 240 sums of products vanish.

As seen the labour will be immensely reduced by those circumstances.

Further a variety of equations may be developed to be used as control formulæ.

Most of the computations for the normal equations have been made by a clerk and with the aid of the calculating machines »Original Odhner» and »Burroughs Addition Machine». They have also been thoroughly and completely checked by help of several control formulæ.

CHAPTER V.

The observational data.

17. The observational data used for the investigation are the proper motions of all stars brighter than the sixth magnitude, as given in Boss' *Preliminary General Catalogue*, in all 4041 stars. As this catalogue is complete as far down as to the magnitude 6.0 our material consists with a few exclusions of all stars in the heavens brighter than this magnitude.

In his above cited work *Studies in stellar statistics II: The motion of the stars* CHARLIER has divided this material in 48 groups as to the position of the stars, and for each region put together the motions in correlation tables. The 48 regions are obtained by dividing the sky into »squares» symmetrically distributed with regard to the equator. The squares have equal area, and are selected according to the following table.

Squares of the northern hemisphere.

Square	Declination between	Right- ascension between	Square	Declination between	Right- ascension between	Square	Declination between	Right- ascension between
C_1	$0^\circ-30^\circ$	$0^\circ-30^\circ$	B_1	$30^\circ-66.^\circ44$	$0^\circ-36^\circ$	A_1	$66.^\circ44-90^\circ$	$0^\circ-180^\circ$
C_2	» »	$30^\circ-60^\circ$	B_2	» »	$36^\circ-72^\circ$	A_2	» »	$180^\circ-360^\circ$
C_3	» »	$60^\circ-90^\circ$	B_3	» »	$72^\circ-108^\circ$			
C_4	» »	$90^\circ-120^\circ$	B_4	» »	$108^\circ-144^\circ$			
C_5	» »	$120^\circ-150^\circ$	B_5	» »	$144^\circ-180^\circ$			
C_6	» »	$150^\circ-180^\circ$	B_6	» »	$180^\circ-216^\circ$			
C_7	» »	$180^\circ-210^\circ$	B_7	» »	$216^\circ-252^\circ$			
C_8	» »	$210^\circ-240^\circ$	B_8	» »	$252^\circ-288^\circ$			
C_9	» »	$240^\circ-270^\circ$	B_9	» »	$288^\circ-324^\circ$			
C_{10}	» »	$270^\circ-300^\circ$	B_{10}	» »	$324^\circ-360^\circ$			
C_{11}	» »	$300^\circ-330^\circ$		» »				
C_{12}	» »	$330^\circ-360^\circ$		» »				

For each square of the northern hemisphere there is one diametrically situated on the the southern hemisphere. As the squares are to be regarded as plane, the proper motions of the stars in such a pair are referred to the same axes of coordinates.

Accordingly the correlation tables of two such diametrically situated squares may be added (when changing the sign of the proper motion in right ascension on the southern hemisphere) and the moments computed for the motions. The moments given in table I are the moments of the apparent motion about the mean of such a double square. In the following I shall always refer to the double-squares as shortly squares, thus by C_1 meaning the double-cone having the solid angle C_1 . Similarly the other double-squares are denoted by the name of its northern component.

Now, one objection may be raised against this arrangement. When, namely, the system of stars as a whole should be affected by a *rotational* motion about some axis, then clearly the »spread» of the motions in our compound squares should be greatened, as the rotation must affect the two components in opposite directions. Consequently we should find our second and fourth moments too great. Indeed, CHARLIER, has found, for the stars here employed, such a rotational motion amounting to $\rho = 0''.0035$ per year, about an axis nearly normal to the plane of the Milky Way*. This rotational motion is, however, small enough to be altogether neglected when taking together the stars in diametrical squares. Assuming, for instance, the number of stars in the two component squares to be equal, we should have to correct ν_{20} by $-\rho^2 \sin^2 \varphi$; ν_{02} by $-\rho^2 \cos^2 \varphi$; ν_{11} by $-\rho^2 \sin \varphi \cos \varphi$. The moments of the third order are not affected at all and by those of the fourth order we should have to correct, for instance, $\nu_{40} - 3\nu_{20}^2$ by $+\rho^4 \sin^4 \varphi$. For φ we have to take the angle between the x -axis of the square and the axis of rotation, and to find the above corrections formulae (68*) (68**) and (68***) can be used. As $\rho = \frac{\omega}{14}$, we see that taking the classbreadth as unity the corrections of the second and fourth characteristics at most amount to 0.0051 and 0.000026, which can be wholly neglected, especially as we have already neglected the corrections of SHEPPARD for the classbreadth, which affect even one higher of the decimal places.

The direction cosines of the centres of gravity of the squares are according to CHARLIER** as put together in table V.

* This is the motion of the node of the invariable plane of the solar system upon the plane of the Galaxy. It is direct and has a period of about 370 million years.

** The direction cosines for the U - and V -axes tabulated by CHARLIER l. c. p. 70 we must take with changement of sign.

TABLE V.
Direction cosines of the squares

	γ_{11}	γ_{12}	γ_{13}	γ_{21}	γ_{32}	γ_{23}	γ_{32}	γ_{33}
A_1	-1.0000	0.0000	0.0000	0.0000	-0.9848	+0.1736	+0.1736	+0.9848
A_2	+1.0000	0.0000	0.0000	0.0000	+0.9848	-0.1736	+0.1736	+0.9848
B_1	-0.3090	-0.6737	+0.6714	+0.9511	-0.2189	+0.2181	+0.7059	+0.7088
B_2	-0.8090	-0.4163	+0.4149	+0.5878	-0.5730	+0.5711	+0.7059	+0.7088
B_3	-1.0000	0.0000	0.0000	0.0000	-0.7083	+0.7059	+0.7059	+0.7083
B_4	-0.8090	+0.4163	-0.4149	-0.5878	-0.5730	+0.5711	+0.7059	+0.7088
B_5	-0.3090	+0.6737	-0.6714	-0.9511	-0.2189	+0.2181	+0.7059	+0.7083
B_6	+0.3090	+0.6737	-0.6714	-0.9511	+0.2189	-0.2181	+0.7059	+0.7083
B_7	+0.8090	+0.4163	-0.4149	-0.5878	+0.5730	-0.5711	+0.7059	+0.7083
B_8	+1.0000	0.0000	0.0000	0.0000	+0.7083	-0.7059	+0.7059	+0.7083
B_9	+0.8090	-0.4163	+0.4149	+0.5878	+0.5730	-0.5711	+0.7059	+0.7083
B_{10}	+0.3090	-0.6737	+0.6714	+0.9511	+0.2189	-0.2181	+0.7059	+0.7083
C_1	-0.2588	-0.2415	+0.9353	+0.9659	-0.0647	+0.2506	+0.9683	+0.2500
C_2	-0.7071	-0.1768	+0.6847	+0.7071	-0.1768	+0.6847	+0.9683	+0.2500
C_3	-0.9659	-0.0647	+0.2506	+0.2588	-0.2415	+0.9353	+0.9683	+0.2500
C_4	-0.9659	+0.0647	-0.2506	-0.2588	-0.2415	+0.9353	+0.9683	+0.2500
C_5	-0.7071	+0.1768	-0.6847	-0.7071	-0.1768	+0.6847	+0.9683	+0.2500
C_6	-0.2588	+0.2415	-0.9353	-0.9659	-0.0647	+0.2506	+0.9683	+0.2500
C_7	+0.2588	+0.2415	-0.9353	-0.9659	+0.0647	-0.2506	+0.9683	+0.2500
C_8	+0.7071	+0.1768	-0.6847	-0.7071	+0.1768	-0.6847	+0.9683	+0.2500
C_9	+0.9659	+0.0647	-0.2506	-0.2588	+0.2415	-0.9353	+0.9683	+0.2500
C_{10}	+0.9659	-0.0647	+0.2506	+0.2588	+0.2415	-0.9353	+0.9683	+0.2500
C_{11}	+0.7071	-0.1768	+0.6847	+0.7071	+0.1768	-0.6847	+0.9683	+0.2500
C_{12}	+0.2588	-0.2415	+0.9353	+0.9659	+0.0647	-0.2506	+0.9683	+0.2500

In the cited work CHARLIER has published the values of $x_0, y_0, v_{20}, v_{11}, v_{02}, v_{30}, v_{03}, v_{40}, v_{04}$ for each of his 48 squares. The values of the remaining moments $v_{21}, v_{12}, v_{31}, v_{22}, v_{13}$ have also been computed and, though unpublished, kindly placed at my disposal. All this work being already done, the most convenient way to get at the moments of the compound squares, is to compute by the formulæ giving the moments of a frequency distribution that is made up of two component distributions added to each other.

We will give those formulæ as they are of a very general interest. But as they depend on algebraical developments of some length we give them without demonstration.

Denoting by index (n) the moments of a northern square, by index (s) the moment of the diametrical square, and by n_1 and n_2 the relative numbers of the stars in the two squares, we have:*

$$v'_{ij} = n_1 v'_{ij}^{(n)} + n_2 v'_{ij}^{(s)},$$

$$n_1 + n_2 = 1.$$

Applying here the formulæ (45), (45*), (45**) we get the equations

* CHARLIERS values for $v_{ij}^{(s)}$ must be taken with negative sign if i is an odd number.

$$(68) \quad \begin{aligned} x_0 &= n_1 x_0^{(n)} + n_2 x_0^{(s)}, \\ y_0 &= n_1 y_0^{(n)} + n_2 y_0^{(s)}, \end{aligned}$$

and putting

$$v_{ij}^{(n)} - v_{ij}^{(s)} = A_{ij}, \quad v_{ij}^{(n)} + v_{ij}^{(s)} = \Sigma_{ij},$$

we find

$$(68^*) \quad \begin{aligned} v_{20} &= n_1 v_{20}^{(n)} + n_2 v_{20}^{(s)} + n_1 n_2 A_{10}^2 \\ v_{11} &= n_1 v_{11}^{(n)} + n_2 v_{11}^{(s)} + n_1 n_2 A_{10} A_{01} \\ v_{02} &= n_1 v_{02}^{(n)} + n_2 v_{02}^{(s)} + n_1 n_2 A_{01}^2 \end{aligned}$$

(68**)

$$\begin{aligned} v_{30} &= n_1 v_{30}^{(n)} + n_2 v_{30}^{(s)} + 3n_1 n_2 A_{20} A_{10} + n_1 n_2 (n_2 - n_1) A_{10}^3 \\ v_{21} &= n_1 v_{21}^{(n)} + n_2 v_{21}^{(s)} + n_1 n_2 A_{20} A_{01} + 2n_1 n_2 A_{11} A_{10} + n_1 n_2 (n_2 - n_1) A_{10}^2 A_{01} \\ v_{12} &= n_1 v_{12}^{(n)} + n_2 v_{12}^{(s)} + n_1 n_2 A_{02} A_{10} + 2n_1 n_2 A_{11} A_{01} + n_1 n_2 (n_2 - n_1) A_{10} A_{01}^2 \\ v_{03} &= n_1 v_{03}^{(n)} + n_2 v_{03}^{(s)} + 3n_1 n_2 A_{02} A_{01} + n_1 n_2 (n_2 - n_1) A_{01}^3 \end{aligned}$$

(68***)

$$\begin{aligned} v_{40} &= n_1 v_{40}^{(n)} + n_2 v_{40}^{(s)} + 4n_1 n_2 A_{30} A_{10} + 3n_1 n_2 (n_2 - n_1) A_{20} A_{10}^2 + \\ &\quad + 3n_1 n_2 \Sigma_{20} A_{10}^2 + n_1 n_2 (1 - 3n_1 n_2) A_{10}^4 \\ v_{31} &= n_1 v_{31}^{(n)} + n_2 v_{31}^{(s)} + 3n_1 n_2 A_{21} A_{10} + n_1 n_2 A_{30} A_{01} \\ &\quad + {}^{3/2} n_1 n_2 (n_2 - n_1) A_{11} A_{10}^2 + {}^{3/2} n_1 n_2 \Sigma_{11} A_{10}^2 \\ &\quad + {}^{3/2} n_1 n_2 (n_2 - n_1) A_{20} A_{10} A_{01} + {}^{3/2} n_1 n_2 \Sigma_{20} A_{10} A_{01} \\ &\quad + n_1 n_2 (1 - 3n_1 n_2) A_{10}^3 A_{01} \\ v_{22} &= n_1 v_{22}^{(n)} + n_2 v_{22}^{(s)} + 2n_1 n_2 A_{21} A_{01} + 2n_1 n_2 A_{12} A_{10} \\ &\quad + 2n_1 n_2 (n_2 - n_1) A_{11} A_{10} A_{01} + 2n_1 n_2 A_{11} A_{10} A_{01} \\ &\quad + {}^{1/2} n_1 n_2 (n_2 - n_1) A_{20} A_{01}^2 + {}^{1/2} n_1 n_2 \Sigma_{20} A_{10}^2 \\ &\quad + {}^{1/2} n_1 n_2 (n_2 - n_1) A_{02} A_{10}^2 + {}^{1/2} n_1 n_2 \Sigma_{02} A_{10}^2 \\ &\quad + n_1 n_2 (1 - 3n_1 n_2) A_{10}^3 A_{01}^2 \\ v_{13} &= n_1 v_{13}^{(n)} + n_2 v_{13}^{(s)} + 3n_1 n_2 A_{12} A_{01} + n_1 n_2 A_{03} A_{10} \\ &\quad + {}^{3/2} n_1 n_2 (n_2 - n_1) A_{11} A_{01}^2 + {}^{3/2} n_1 n_2 \Sigma_{11} A_{01}^2 \\ &\quad + {}^{3/2} n_1 n_2 (n_2 - n_1) A_{02} A_{10} A_{01} + {}^{3/2} n_1 n_2 \Sigma_{01} A_{10} A_{01} \\ &\quad + n_1 n_2 (1 - 3n_1 n_2) A_{10} A_{01}^3 \\ v_{04} &= n_1 v_{04}^{(n)} + n_2 v_{04}^{(s)} + 4n_1 n_2 A_{03} A_{01} + 3n_1 n_2 (n_2 - n_1) A_{02} A_{01}^2 + \\ &\quad + 3n_1 n_2 \Sigma_{01} A_{01}^2 + n_1 n_2 (1 - 3n_1 n_2) A_{01}^4. \end{aligned}$$

In table I we give the moments and number of stars N for each square. The unit is one class breadth = 0.''05 per year.

TABLE I.
Moments of the apparent proper-motion.

	N	x_0	y_0	v_{20}	v_{02}	v_{11}
A_1	144	+ 0.180	- 0.625	+ 1.245	+ 2.504	- 0.589
A_2	159	+ 0.047	+ 0.871	+ 0.739	+ 3.038	+ 0.270
B_1	208	+ 0.697	- 0.573	+ 2.542	+ 1.124	- 0.164
B_2	216	+ 0.509	- 0.745	+ 1.111	+ 1.103	- 0.591
B_3	183	+ 0.161	- 0.969	+ 0.651	+ 2.042	+ 0.065
B_4	123	- 0.492	- 0.671	+ 1.456	+ 1.458	+ 0.594
B_5	130	- 0.992	- 0.638	+ 3.110	+ 2.411	+ 0.233
B_6	106	- 0.707	- 0.245	+ 2.674	+ 1.265	- 0.655
B_7	131	- 0.744	+ 0.225	+ 3.452	+ 3.680	- 1.609
B_8	178	+ 0.180	+ 0.298	+ 1.353	+ 3.172	+ 0.457
B_9	283	+ 0.496	- 0.019	+ 0.922	+ 1.508	+ 0.627
B_{10}	214	+ 0.658	- 0.153	+ 1.855	+ 0.918	+ 0.614
C_1	123	+ 0.744	- 0.663	+ 3.696	+ 1.583	+ 0.167
C_2	178	+ 0.781	- 0.775	+ 2.900	+ 1.458	+ 0.357
C_3	227	+ 0.492	- 0.703	+ 1.445	+ 0.834	+ 0.136
C_4	207	- 0.249	- 0.698	+ 1.240	+ 1.598	- 0.158
C_5	141	- 0.752	- 0.961	+ 2.093	+ 2.489	- 0.057
C_6	136	- 0.838	- 0.632	+ 4.681	+ 1.218	- 0.450
C_7	114	- 0.789	- 0.403	+ 2.890	+ 2.334	- 0.753
C_8	132	- 0.834	- 0.576	+ 3.946	+ 1.584	+ 0.091
C_9	191	- 0.249	- 0.322	+ 1.361	+ 1.617	- 0.458
C_{10}	255	+ 0.320	- 0.198	+ 1.065	+ 1.513	- 0.151
C_{11}	135	+ 0.622	- 0.474	+ 2.522	+ 1.601	+ 0.093
C_{12}	107	+ 1.083	- 0.164	+ 3.430	+ 2.018	+ 0.135

	v_{80}	v_{21}	v_{12}	v_{08}	v_{40}	v_{81}	v_{22}	v_{18}	v_{04}
A_1	+ 1.449	- 1.035	+ 1.583	- 2.607	+ 10.175	- 8.007	+ 13.905	- 19.170	+ 40.748
A_2	- 0.602	- 0.296	- 0.509	+ 0.790	+ 3.487	+ 0.647	+ 3.284	- 0.053	+ 67.591
B_1	+ 5.673	- 1.130	- 1.721	- 0.519	+ 47.708	- 3.454	+ 12.783	- 1.489	+ 28.017
B_2	+ 1.978	- 1.309	+ 1.949	- 2.892	+ 11.597	- 4.999	+ 6.444	- 9.655	+ 14.363
B_3	+ 0.001	- 0.508	- 0.501	- 2.701	+ 1.810	+ 0.657	+ 2.867	+ 4.681	+ 25.566
B_4	- 0.319	+ 0.135	- 0.256	- 1.318	+ 10.048	+ 3.205	+ 4.959	+ 6.358	+ 13.054
B_5	- 4.640	- 0.327	- 1.408	- 5.453	+ 49.207	+ 0.842	+ 9.913	+ 3.328	+ 50.024
B_6	- 0.685	+ 0.517	- 0.757	- 0.550	+ 18.666	- 4.239	+ 4.462	- 3.858	+ 8.322
B_7	- 9.381	+ 4.349	- 4.337	+ 1.548	+ 75.459	- 34.446	+ 41.209	- 43.322	+ 95.552
B_8	+ 3.069	+ 0.490	+ 0.749	+ 6.583	+ 19.441	+ 3.584	+ 7.796	+ 11.144	+ 78.108
B_9	+ 1.965	+ 1.787	+ 2.338	+ 1.881	+ 12.797	+ 11.619	+ 14.374	+ 13.239	+ 27.396
B_{10}	+ 1.146	- 0.903	+ 0.297	- 0.21	+ 30.088	+ 6.106	+ 5.791	+ 2.905	+ 6.116
C_1	- 5.014	- 3.241	- 0.708	- 2.410	+ 73.479	+ 17.421	+ 15.325	+ 5.540	+ 17.035
C_2	+ 2.359	- 0.402	+ 1.678	+ 1.423	+ 56.330	+ 19.632	+ 28.677	+ 24.273	+ 37.162
C_3	- 0.269	- 0.693	- 0.383	- 1.687	+ 13.889	+ 2.878	+ 2.747	+ 0.187	+ 7.524
C_4	+ 1.299	- 0.718	+ 1.655	- 4.167	+ 16.160	- 2.840	+ 13.589	- 7.377	+ 27.814
C_5	- 3.785	- 0.867	- 0.205	- 9.253	+ 27.002	- 4.073	+ 20.891	- 1.383	+ 64.636
C_6	- 2.383	- 4.266	+ 0.673	- 1.582	+ 120.926	- 24.112	+ 29.440	- 3.928	+ 19.736
C_7	- 2.545	- 0.663	+ 0.379	- 2.313	+ 40.171	- 11.987	+ 15.130	- 10.078	+ 52.419
C_8	- 5.159	- 1.604	- 0.650	- 1.806	+ 70.673	- 2.104	+ 7.860	+ 0.855	+ 9.693
C_9	+ 0.308	- 1.474	+ 2.288	- 2.330	+ 21.036	- 12.078	+ 17.465	- 16.015	+ 27.067
C_{10}	+ 1.626	- 1.080	+ 1.077	- 3.562	+ 10.048	- 2.966	+ 6.800	- 6.056	+ 27.586
C_{11}	+ 2.523	- 0.804	- 0.169	- 2.307	+ 44.577	- 9.438	+ 15.803	- 6.594	+ 20.295
C_{12}	+ 5.478	+ 0.889	+ 0.085	- 1.743	+ 44.249	+ 6.691	+ 8.688	+ 2.491	+ 27.599

CHAPTER VI.

The normal equations and the characteristics as referred to
the system of the equator.

18. At the end of this memoir we in tables II, III and IV give the values of the coefficients of the equations (58), (64) and (66) for each square. From these the normal equations are computed.

For the second characteristics we obtain the normal equations

$$\begin{aligned}
 x_1 \cdot 11.9312 + y_1 \cdot 2.3492 + z_1 \cdot 1.6016 &= 27.1655 q' - 3.0432 \\
 x_1 \cdot 2.3492 + y_1 \cdot 11.8124 + z_1 \cdot 1.6600 &= 53.5855 q' - 9.2538 \\
 x_1 \cdot 1.6016 + y_1 \cdot 1.6600 + z_1 \cdot 13.0340 &= 33.5440 q' - 5.4571 \\
 p_1 \cdot 17.1058 &= -4.0998 q' + 0.5636 \\
 q_1 \cdot 14.7558 &= -0.5154 q' + 6.6289 \\
 r_1 \cdot 14.5844 &= +3.1957 q' - 0.5963,
 \end{aligned}$$

which give

$$\begin{aligned}
 \mathfrak{D}_1^2 N''_{200} &= x_1 = +1.2281 q' - 0.0697 \\
 \mathfrak{D}_1^2 N''_{020} &= y_1 = +4.0236 q' - 0.7249 \\
 \mathfrak{D}_1^2 N''_{002} &= z_1 = +1.9110 q' - 0.3177 \\
 \mathfrak{D}_1^2 N''_{110} &= p_1 = -0.2397 q' + 0.0330 \\
 \mathfrak{D}_1^2 N''_{011} &= q_1 = -0.0349 q' + 0.4492 \\
 \mathfrak{D}_1^2 N''_{101} &= r_1 = +0.2191 q' - 0.0409.
 \end{aligned}$$

To obtain, from these values, the axes of the ellipsoid we must form the cubic (61*). We get

$$\begin{aligned}
 s^3 - s^2 (7.1627 q' - 1.1123) + s (14.8707 q'^2 - 4.2923 q' + 0.0983) - \\
 - (9.1421 q'^3 - 3.6607 q'^2 + 0.2064 q' - 0.0017) = 0.
 \end{aligned}$$

This equation we cannot solve without using numerical values of q' . Hence we give in table VI the roots for four different values of q' . Computing then the vertices from equations (63) and the axes from (61**) the table VII is obtained. The unit is the *class breadth* which is very near equal to the velocity of the sun.

The values of q' are chosen rather arbitrarily. The value $q' = 0.6205$ is the one used by CHARLIER.

TABLE VI.

The roots of the cubic.

q'	s_1	s_2	s_3
1.0000	3.4057	1.6036	1.0412
0.8422	2.7906	1.2778	0.8517
0.6205	1.9462	0.8234	0.5626
0.4500	1.3270	0.5177	0.2663

TABLE VII.

The Axes and the Vertices of the Velocity-ellipsoid for different values of q' .

q'	$\vartheta_1 \tau_1$	α_1	δ_1	$\vartheta_1 \tau_2$	α_2	δ_2	$\vartheta_1 \tau_3$	α_3	δ_3	Δ_3
1.00	1.8454	274° 3	—12° 4	1.2663	339° 1	62° 5	1.0204	189° 9	24° 1	4°
0.84	1.6705	274° 1	—15° 8	1.1304	338° 3	57° 9	0.9229	192° 2	27° 5	1°
0.62	1.3950	273° 6	—21° 3	0.9074	342° 1	43° 3	0.7501	202° 0	39° 0	16°
0.45	1.1520	272° 9	—28° 7	0.7195	351°	21°	0.5160	230° 6	53° 5	37°

The pole of the Milky way is according to KOBOLD

$$\alpha = 191^\circ 2$$

$$\delta = 28^\circ 0.$$

The last column of table VII contains *the distance of the smallest axis from this pole*. The distance will be a minimum for q' somewhat smaller than 0.84, and it will then not amount to even 1°. Further we see that when q' decreases below 0.84 this axis will lie more and more distant from the pole, being for $q' = 0.45$ already 37° from it.

Accordingly we see that *for q' between 1.0 and 0.7 the system of the axes of the velocity ellipsoid can be regarded as a galactic system*.

As it has been found by GYLLENBERG from the radial velocities that the velocity ellipsoid has two axes in the plane of the Milky Way we already here may conclude that *q' cannot well have a value below 0.7*. Compare, however, p. 66.

The value $q' = 1$ gives the ellipsoid of the *apparent* motions. We call it the apparent ellipsoid. We find, that using as unity the second of arc we have:

The Apparent Ellipsoid

$$\begin{array}{lll} \text{Axes: } \sigma_1 = 0''.0923 & \text{Vertex: } \alpha_1 = 274^\circ 3 & \delta_1 = -12^\circ 4 \\ \sigma_2 = 0''.0633 & \alpha_2 = 339^\circ 1 & \delta_2 = +62^\circ 5 \\ \sigma_3 = 0''.0510 & \alpha_3 = 189^\circ 9 & \delta_3 = +24^\circ 1. \end{array}$$

The Apex of the solar motion referred to the stars here used has been computed by CHARLIER. A recomputation performed by me of course gave the same result. It is:

$$\begin{aligned}\text{Solar motion and Apex. } \vartheta_1 S &= 1.0275 = 0.''05138 \\ A &= 272.^{\circ}7 \\ D &= 31.^{\circ}6.\end{aligned}$$

Taking for the value of S the solar velocity found by GYLLENBERG from 1114 radial velocities, or

$$S = 19.4 \text{ km. per sec.}$$

we obtain

$$(1) \quad \vartheta_1 = 0.0529,$$

which value is to be used to get our attributes of motion expressed in the km. per sec. as unity.

Generally it will be more adequate in discussing stellar movements, to use as unity of length and time the *Siriometer* $= 10^6$ times the mean distance of the Sun and the *Stellar year* $= 10^6$ years*. Thus the velocity one Siriometer per stellar year is the same as one mean distance of the Sun per year.

In those units we have

$$S = 4.103,$$

and obtain

$$(2) \quad \vartheta_1 = 0.2504.$$

Multiplying with the classbreadth we find the mean parallax of the stars brighter than 6.0

$$\pi_1 = 0.''01252.$$

Using the value (1) of ϑ_1 we get the table VIII for the axes. In this table also the excentricities are contained. Thus e_{12} is the excentricity in the plane

VIII.

q'	σ_1 km	σ_2 km	σ_3 km	e_{12}	e_{13}
1.0000	34.88	23.93	19.29	0.73	0.83
0.8422	31.57	21.36	17.44	0.74	0.83
0.6205	26.36	17.15	14.18	0.75	0.83
0.4500	21.77	13.60	9.75	0.78	0.89

of σ_1 and σ_2 , a. s. f. Of course the excentricities are independent of ϑ_1 , but we here see also, that they are practically independent of q' . Thus we may state that *the form of the ellipsoid is the same whatever (in any way plausible) values*

* Those units have been adopted by CHARLIER in his lectures on stellar statistics.

of q' or ϑ_1 be used. But it will be affected if we correct for a ϑ_1 varying with galactic latitude. In the main we characterize the velocity ellipsoid as follows: *The velocity ellipsoid is an ellipsoid with three unequal axes. The shortest axis is directed toward the galactic pole and the longest one near the galactic point $\alpha = 273^\circ$ $\delta = -15^\circ$. The excentricities of the ellipsoid in sections transverse to the three axes are $e_{12} = 0.75$, $e_{13} = 0.84$, $e_{23} = 0.57$. If ϑ_1 varies with galactic latitude those excentricities shall be slightly corrected, else they are highly independent of the assumptions regarding the density- and luminosity-laws.*

The most interesting feature of the ellipsoid as found here from the proper motions is that *it has not the same form as the ellipsoid of the radial motions**. Indeed, the radial motions produce an ellipsoid having the *medium* axes directed toward the pole of the Galaxy. GYLLENBERG has found that for all stars brighter than the magnitude 4.9 we have the axes **

$$\begin{aligned}\sigma_1 &= 19.7 \text{ km.} \\ \sigma_2 &= 12.5 \text{ »} \\ \sigma_3 &= 16.9 \text{ »}\end{aligned}$$

and this form is qualitatively the same for all spectral classes save the *B*- and *M*-type, of which there is procentually very few among the stars brighter than 6.0.

Taking the mean of the square axes for the types *A*, *F* and *G* GYLLENBERG's values will give:

$$\begin{aligned}\sigma_1 &= 21.2 \text{ km.} \\ \sigma_2 &= 13.3 \text{ »} \\ \sigma_3 &= 17.5 \text{ »}\end{aligned}$$

The transgalactic axis σ_3 lies within 6° of the pole of the Milky Way and the longest galactic axis points towards $\alpha = 273^\circ$ $\delta = -12^\circ$.

Now, of course, those values are not directly comparable with our values for different q' as the direction of the axes also varies with q' . The more careful comparison shall be saved till a following chapter, where the characteristics referred to a fixed galactic system of coordinates can be discussed. Already here we can, however, note some circumstances. It will be seen, for instance, that *the ratio of the two galactic axes is the same in the ellipsoids of proper motion and radial motion*. Further that the galactic axes will be of the same magnitude in the two ellipsoids if q' is taken about 0.45, but that the transgalactic axis requires a q' of the order of magnitude of 0.80. The question is now which of the two sorts of axes shall be compared? Against using the galactic axes speaks the exceedingly low value of q' for which the vertex positions do not agree. Against the other axis, however, speaks the fact that it is determined from galactic stars when using the proper motion, but from non-galactic stars when employing the radial velocities. But, anyway, the last objection seems to be the one of least importance.

* This fact was first noted by GYLLENBERG. Compare Medd. Nr. 59.

** Those figures are as yet unpublished. They have been kindly placed at my disposal.

There seems to be principally three possible explanations of this difference between the results of radial- and proper-motions*. They are:

1:0. If the stars out of the Galaxy have greater spread of motion than the galactic stars.

2:0. If the mean parallax ϑ_1 is greater near the pole of the Milky Way than in its plane.

3:0. If the tangential components of motion parallel to the galactic plane generally predominate over the radial components.

The second explanation is decidedly the most simple and plausible one, but it has the fault that the difference to be explained is too great. Indeed, it would for $q' = 0.75$ require a ratio of ϑ_1 out of the Galaxy to ϑ_1 in the Galaxy amounting to about 2:1. Now for the stars brighter than the magnitude 6.0 it is not probable, that the boundary of the stellar system being further away in the Galaxy than at the pole, should have any effect on the mean parallax. Indeed, the variation must principally come forth on account of the predominance of the early spectral class stars in the Galaxy compared to the polar regions. The variation then cannot be anywhere so great. In fact, as will be seen in a following chapter the proper motions themselves make probable a ratio amounting to about 1.2. As a summary it is only the more emphasized of how great an importance it is to study the variation in ϑ_1 and correct the axes for it. In a following chapter we will, in fact, more carefully look into the thing, and we will there also procure a method of correcting the characteristics for this variation.

To explain the difference recourse must also be taken to one of the other hypotheses or both. To come to a decision it will perhaps be most expedient to compute the ellipsoid separately from stars in the galactic and in the polar regions. For the present I will, however, not go further into these questions, hoping to find an opportunity to do so another time.

19. The normal equations for the characteristics of the third order take the following simple form:

$$- 3.8512 B''_{111} \vartheta_1^3 = + 1.3282 q'^3 - 0.5897 q'(1 - q'^2) + 0.1423(2 - 3q' + q'^3)$$

$$\begin{array}{rclclcl} 13.9552 B''_{300} \vartheta_1^3 & + & 0.1084 B''_{120} \vartheta_1^3 & - & 1.6878 B''_{102} \vartheta_1^3 & = & + 0.5184 q'^3 + 0.6877 q'(1 - q'^2) - 0.0342(2 - 3q' + q'^3) \\ 0.1084 & & - 5.9383 & & - 0.6209 & = & - 1.5020 + 0.9243 - 0.1210 \\ 1.6878 & & - 0.6209 & & - 7.4710 & = & - 0.5289 + 0.1477 - 0.1463 \\ 13.7772 B''_{030} \vartheta_1^3 & + & 0.1084 B''_{210} \vartheta_1^3 & - & 1.7438 B''_{012} \vartheta_1^3 & = & + 10.7940 q'^3 + 20.1610 q'(1 - q'^2) + 1.6458(2 - 3q' + q'^3) \\ 0.1084 & & - 5.9902 & & - 0.5624 & = & + 0.9566 - 3.6172 + 0.0976 \\ 1.7438 & & - 0.1624 & & - 7.4710 & = & + 2.2416 - 7.2754 + 1.5002 \\ 11.1286 B''_{003} \vartheta_1^3 & - & 0.9527 B''_{201} \vartheta_1^3 & - & 0.9545 B''_{021} \vartheta_1^3 & = & - 4.9453 q'^3 + 5.7262 q'(1 - q'^2) - 0.5677(2 - 3q' + q'^3) \\ 0.9527 & & - 8.2352 & & - 0.5904 & = & - 3.6788 + 3.5266 - 0.3886 \\ 0.9545 & & - 0.5904 & & - 8.2320 & = & - 3.1457 + 5.4148 - 1.7464 \end{array}$$

* On the whole all possible explanations of the difference between the results deduced from the cross motions and from the radial motions must be sought either in a variation of the parameters ϑ_1 and q' in different regions of the sky, or in the circumstance that the distribution of the velocities is not strictly the same in all parts of the heavens.

Writing generally (for $i + j + k = 3$)

$$\mathfrak{P}_1^3 B_{ijk} = H_{ijk}^{(1)} q'^3 + H_{ijk}^{(2)} q'(1 - q'^2) + H_{ijk}^{(3)} (2 - 3q' + q'^3)$$

we obtain as solutions the following table.

Characteristics of the third order referred to the Equatoreal System.

	$H_{ijk}^{(1)}$	$H_{ijk}^{(2)}$	$H_{ijk}^{(3)}$
$\mathfrak{P}_1^3 B''_{800}$	-0.0425	-0.0511	+0.00042
$\mathfrak{P}_1^3 B''_{120}$	+0.2450	-0.1570	+0.01845
$\mathfrak{P}_1^3 B''_{102}$	+0.0598	+0.0049	+0.01781
$\mathfrak{P}_1^3 B''_{080}$	-0.7711	+1.3911	-0.09693
$\mathfrak{P}_1^3 B''_{210}$	-0.1633	+0.5714	-0.00124
$\mathfrak{P}_1^3 B''_{012}$	-0.1018	+0.6061	-0.17810
$\mathfrak{P}_1^3 B''_{003}$	+0.3852	-0.4757	+0.03083
$\mathfrak{P}_1^3 B''_{201}$	+0.3806	-0.3370	+0.02888
$\mathfrak{P}_1^3 B''_{021}$	+0.3102	-0.5881	+0.20650
$\mathfrak{P}_1^3 B''_{111}$	-0.3449	+0.1531	-0.03695

20. Finally the normal equations for the characteristics of the fourth order will be of 4 kinds. We have

3 equations of the form

$$\mathfrak{P}_1^4 B''_{130} h_1 + \mathfrak{P}_1^4 B''_{310} h_2 + \mathfrak{P}_1^4 B''_{112} h_3 = h_4 q'^6 + h_5 q'^3(1 - q'^3) + h_6 q'(2 - q' - 2q'^3 + q'^5) + h_7 q'^2 + h_8(6 - 12q' + 3q'^2 + 4q'^3 - q'^6),$$

as given by the table

h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
5.1872	-0.1910	+0.4208	-4.6422	+2.1772	-0.9482	+1.5501	+0.0623
-0.1910	+5.3017	+0.3659	-1.3703	+0.1140	-0.2482	-0.0091	+0.0101
+0.4208	+0.3659	+2.6662	-1.6690	+1.5274	-0.2911	+0.6524	+0.0266

3 equations of the form

$$\mathfrak{P}_1^4 B''_{103} k_1 + \mathfrak{P}_1^4 B''_{301} k_2 + \mathfrak{P}_1^4 B''_{121} k_3 = k_4 q'^6 + k_5 q'^3(1 - q'^3) + k_6 q'(2 - q' - 2q'^2 + q'^5) + k_7 q'^2 + k_8(6 - 12q' + 3q'^2 + 4q'^3 - q'^6),$$

given in the table,

k_1	k_2	k_3	k_4	k_5	k_6	k_7	k_8
+6.1888	+0.9037	+0.3234	-5.4631	+0.2298	-0.1220	+0.5826	-0.0073
+0.9037	+7.8612	+0.2104	-2.2731	-0.1651	+0.5974	-0.7957	-0.0357
+0.3234	+0.2101	+2.5169	-2.2637	-0.4414	+0.1218	+0.5936	-0.0326

3 equations of the form

$$\mathfrak{D}_1^4 B''_{013} l_1 + \mathfrak{D}_1^4 B''_{031} l_2 + \mathfrak{D}_1^4 B''_{211} l_3 = l_4 q'^6 + l_5 q'^3(1 - q'^3) + l_6 q'(2 - q' - 2q'^2 + q'^5) + l_7 q'^2 + l_8(6 - 12q' + 3q'^2 + 4q'^3 - q'^6),$$

given in the table

l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8
+ 6.1830	+ 0.9657	+ 0.3210	+ 8.3133	+ 2.7977	- 2.9961	- 2.7019	+ 0.2773
+ 0.9657	+ 7.8570	+ 0.2080	+ 16.5260	+ 4.4352	- 5.2532	- 6.0618	+ 0.5065
+ 0.3210	+ 0.2080	+ 2.5210	+ 5.0679	+ 1.0559	- 1.2143	- 1.3284	+ 0.0551

finally 6 equations of the form

$$\begin{aligned} \mathfrak{D}_1^4 B''_{400} m_1 + \mathfrak{D}_1^4 B''_{040} m_2 + \mathfrak{D}_1^4 B''_{004} m_3 + \mathfrak{D}_1^4 B''_{220} m_4 + \mathfrak{D}_1^4 B''_{202} m_5 + \mathfrak{D}_1^4 B''_{022} m_6 = \\ = m_7 q'^6 + m_8 q'^3(1 - q'^3) + m_9(2 - q' - 2q'^2 + q'^5) q' + m_{10} q'^2 + \\ + m_{11}(6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \end{aligned}$$

and given by the table

m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}
+ 14.7440	+ 0.0524	+ 0.2500	- 0.2337	+ 2.0474	+ 0.5914	+ 13.8829	- 0.9798	+ 0.7737	- 4.1455	- 0.0410
+ 0.0524	+ 14.5154	+ 0.2520	- 0.2341	+ 0.5911	+ 2.1024	+ 48.8587	- 10.3444	+ 9.8267	- 23.0590	- 0.4800
+ 0.2500	+ 0.2520	+ 9.8906	+ 0.0834	+ 0.6196	+ 0.6190	+ 15.9906	- 3.2800	+ 1.8801	- 5.1859	- 0.1000
- 0.2337	- 0.2342	+ 0.0634	+ 4.1712	+ 0.4981	+ 0.4392	+ 10.5492	- 0.8894	+ 1.1292	- 4.7261	- 0.0336
+ 2.0474	+ 0.5911	+ 0.6196	+ 0.4981	+ 6.0904	+ 0.7602	+ 24.3770	- 2.2405	+ 1.7239	- 6.0074	- 0.1529
+ 0.5914	+ 2.1024	+ 0.6190	+ 0.4392	+ 0.7602	+ 6.0908	+ 36.7593	- 5.0750	+ 4.6168	- 13.5552	- 0.4809

Solving, we obtain as solutions

$$\begin{aligned} \mathfrak{D}_1^4 B'_{ijk} = H_{ijk}^{(1)} q'^6 + H_{ijk}^{(2)} q'^3(1 - q'^3) + H_{ijk}^{(3)}(2 - q' - 2q'^2 + q'^5) q' + \\ + H_{ijk}^{(4)} q'^2 + H_{ijk}^{(5)}(6 - 12q' + 3q'^2 + 4q'^3 - q'^6) \end{aligned}$$

and we find:

Characteristics of the fourth order referred to the Equatoreal System.

	$H_{ijk}^{(1)}$	$H_{ijk}^{(2)}$	$H_{ijk}^{(3)}$	$H_{ijk}^{(4)}$	$H_{ijk}^{(5)}$
$\mathfrak{P}_1^4 B''_{400}$	+ 0.3741	— 0.0120	+ 0.0136	— 0.1551	— 0.0000
$\mathfrak{P}_1^4 B''_{040}$	+ 2.6097	— 0.6223	+ 0.6005	— 1.3493	— 0.0229
$\mathfrak{P}_1^4 B''_{004}$	+ 1.0677	— 0.2672	+ 0.1332	— 0.3482	— 0.0045
$\mathfrak{P}_1^4 B''_{220}$	+ 1.8665	— 0.1619	+ 0.2347	— 0.9858	— 0.0000
$\mathfrak{P}_1^4 B''_{202}$	+ 2.7998	— 0.1933	+ 0.1246	— 0.4903	— 0.0149
$\mathfrak{P}_1^4 B''_{022}$	+ 4.5055	— 0.5542	+ 0.5034	— 1.5770	— 0.0656
$\mathfrak{P}_1^4 B''_{180}$	— 0.8677	+ 0.3781	— 0.1783	+ 0.2323	+ 0.0114
$\mathfrak{P}_1^4 B''_{810}$	— 0.2584	— 0.0003	— 0.0479	— 0.0054	+ 0.0018
$\mathfrak{P}_1^4 B'_{112}$	— 0.4536	+ 0.5132	— 0.0767	+ 0.2009	+ 0.0794
$\mathfrak{P}_1^4 B''_{108}$	— 0.8165	+ 0.0501	— 0.0345	+ 0.1007	+ 0.0001
$\mathfrak{P}_1^4 B''_{301}$	— 0.1682	— 0.0223	+ 0.0789	— 0.1198	— 0.0042
$\mathfrak{P}_1^4 B''_{121}$	— 0.7805	— 0.1799	+ 0.0403	+ 0.2329	— 0.0126
$\mathfrak{P}_1^4 B''_{018}$	+ 0.9517	+ 0.3513	— 0.3689	— 0.3018	+ 0.0349
$\mathfrak{P}_1^4 B''_{081}$	+ 1.9406	+ 0.5380	— 0.6131	— 0.7231	+ 0.0599
$\mathfrak{P}_1^4 B''_{211}$	+ 1.7290	+ 0.3297	— 0.3841	— 0.4289	+ 0.0145

CHAPTER VII.

The characteristics referred to a galactic system of coordinates.

21. For many obvious reasons it is preferable to have the characteristics of the motion referred to a system of coordinates having one axis directed toward the galactic pole. Accordingly we will perform the transformation of all the characteristics to the system III defined in § 8. The transformation is obtained algebraically by the equations (57) and permutations for the second moments, and by (41) or (43) for the higher characteristics, exchanging the direction cosines γ_{ij} against α_{ij} . The reason why the characteristics were not at once transformed to the system III from each square, obviously lies in the advantage gained on account of the symmetry to the equator. Thus the way *via* the system II is really a short cut, besides giving the vertices in equatoreal coordinates.

As the principal vertex (vertex of the longest axis) lies in the Milky Way we naturally take the $U^{(s)}$ axis in that direction. Originally it was my intention to take the vertex system for $q' = 0.8$ as the system III*. By a calamity this vertex-system was slightly miscalculated and consequently the system III does not coincide with the vertices of $q' = 0.8$. As this was not discovered before the transformations of the characteristics were nearly completed I did not think it worth while to alter the system. Besides the difference is very small, any of the axes lying within 4° of the vertices mentioned and the $W^{(s)}$ axis lying within $2^\circ.5$ of the galactic pole of Kobold.

The system III is now defined as follows

System III: Direction of the	$U^{(s)}$ -axis	$\alpha = 270^\circ.0$	$\delta = -15^\circ.0$
	$V^{(s)}$ -axis	$\alpha = 335^\circ.7$	$\delta = +56^\circ.9$
	$W^{(s)}$ -axis	$\alpha = 188^\circ.5$	$\delta = +28^\circ.7$

* Unfortunately I was not aware at the time of the fact that GYLLENBERG had already made use of a similar system. His system could as well have been used, and it is to deplore that there is a difference, though very small.

and the direction cosines are

$$\begin{array}{lll}
 \alpha_{11} = 0.0000 & \alpha_{12} = + 0.4978 & \alpha_{13} = - 0.8673 \\
 \alpha_{21} = - 0.9658 & \alpha_{22} = - 0.2244 & \alpha_{23} = - 0.1289 \\
 \alpha_{31} = - 0.2588 & \alpha_{32} = + 0.8327 & \alpha_{33} = + 0.4807.
 \end{array}$$

Space doesn't permit a reproduction of the numerical equations of transformation. Of course they are of a character to give the characteristics sought explicitly and linearly expressed in terms of the characteristics (of the same order) in the equatorial system. It will suffice to know that the transformations have been thoroughly and painfully checked.

We obtain for the moments of the second order the expressions

$$\begin{array}{ll}
 \mathfrak{P}_1^2 N_{200}^{(s)} = 3.8629 q' - 0.4729 & \mathfrak{P}_1^2 N_{110}^{(s)} = 0.5710 q' - 0.4362 \\
 \mathfrak{P}_1^2 N_{020}^{(s)} = 2.0975 q' - 0.4871 & \mathfrak{P}_1^2 N_{101}^{(s)} = 0.1265 q' - 0.2258 \\
 \mathfrak{P}_1^2 N_{002}^{(s)} = 1.2002 q' - 0.1520 & \mathfrak{P}_1^2 N_{011}^{(s)} = 0.2252 q' - 0.2314
 \end{array}$$

and using the notations of the preceding chapter:

Characteristics of the third order referred to the Galactic System.

	$H_{ijk}^{(1)}$	$H_{ijk}^{(2)}$	$H_{ijk}^{(3)}$
$\mathfrak{P}_1^3 B_{300}^{(s)}$	+ 0.6201	- 1.1441	+ 0.0485
$\mathfrak{P}_1^3 B_{210}^{(s)}$	+ 0.7835	- 1.1464	+ 0.1870
$\mathfrak{P}_1^3 B_{120}^{(s)}$	+ 0.2577	- 0.7289	+ 0.1854
$\mathfrak{P}_1^3 B_{030}^{(s)}$	+ 0.4075	- 0.5166	+ 0.0720
$\mathfrak{P}_1^3 B_{021}^{(s)}$	+ 0.1923	- 0.3358	+ 0.0723
$\mathfrak{P}_1^3 B_{012}^{(s)}$	+ 0.2065	- 0.5225	+ 0.0450
$\mathfrak{P}_1^3 B_{003}^{(s)}$	+ 0.2342	- 0.2248	+ 0.0189
$\mathfrak{P}_1^3 B_{102}^{(s)}$	- 0.1002	- 0.4291	+ 0.0384
$\mathfrak{P}_1^3 B_{201}^{(s)}$	+ 0.2817	- 0.5345	+ 0.0681
$\mathfrak{P}_1^3 B_{111}^{(s)}$	+ 0.1894	- 0.2258	+ 0.2221

Characteristics of the fourth order referred to the Galactic System.

	$H_{ijk}^{(1)}$	$H_{ijk}^{(2)}$	$H_{ijk}^{(3)}$	$H_{ijk}^{(4)}$	$H_{ijk}^{(5)}$
$\vartheta_1^4 B_{400}^{(s)}$	+ 3.0254	— 0.4460	+ 0.4055	— 1.4478	— 0.0093
$\vartheta_1^4 B_{310}^{(s)}$	— 0.4380	— 0.8189	+ 0.7876	— 0.1650	— 0.0497
$\vartheta_1^4 B_{220}^{(s)}$	+ 3.1079	— 0.6933	+ 0.6480	— 1.2333	— 0.0419
$\vartheta_1^4 B_{130}^{(s)}$	— 0.0254	— 0.5198	+ 0.4640	— 0.1060	— 0.0666
$\vartheta_1^4 B_{040}^{(s)}$	+ 0.7039	— 0.2093	+ 0.1781	— 0.2347	— 0.0076
$\vartheta_1^4 B_{331}^{(s)}$	+ 0.1450	— 0.3332	+ 0.1710	— 0.1180	— 0.0133
$\vartheta_1^4 B_{222}^{(s)}$	+ 1.5975	— 0.3353	+ 0.3259	— 0.5967	— 0.0048
$\vartheta_1^4 B_{013}^{(s)}$	+ 1.2625	— 0.2097	+ 0.1897	— 0.1444	— 0.0080
$\vartheta_1^4 B_{004}^{(s)}$	+ 0.3021	— 0.0627	+ 0.0373	— 0.1538	+ 0.0005
$\vartheta_1^4 B_{103}^{(s)}$	— 0.0197	— 0.0989	+ 0.2084	— 0.1394	+ 0.0004
$\vartheta_1^4 B_{202}^{(s)}$	+ 2.8035	— 0.1891	+ 0.2315	— 1.1692	— 0.0232
$\vartheta_1^4 B_{301}^{(s)}$	— 1.4198	— 0.0882	+ 0.2735	+ 0.2802	— 0.0090
$\vartheta_1^4 B_{211}^{(s)}$	+ 1.2024	— 0.5334	+ 0.4709	— 0.2940	— 0.0697
$\vartheta_1^4 B_{121}^{(s)}$	+ 1.0136	— 0.1207	+ 0.2691	— 0.1859	— 0.0489
$\vartheta_1^4 B_{112}^{(s)}$	— 0.4937	+ 0.0763	+ 0.3475	+ 0.1073	+ 0.0024

22. We now shall tabulate the characteristics for the series of different values of q' used in the preceding chapter. For the higher characteristics we, however, find it superfluous to take along the value $q' = 0.45$. The characteristics then are generally greater than for $q' = 0.6205$. The unit is as before the classbreadth. If we put $\vartheta_1 = 20$ we obtain the characteristics expressed in seconds of arc. We find the characteristics as given in tables IX, X, XI.

TABLE IX.

q'	1.0000	0.8422	0.6205	0.4500
$\vartheta_1^2 N_{200}^{(s)}$	+ 3.3900	+ 2.7004	+ 1.9240	+ 1.2654
$\vartheta_1^2 N_{020}^{(s)}$	+ 1.6104	+ 1.2794	+ 0.8143	+ 0.4567
$\vartheta_1^2 N_{002}^{(s)}$	+ 1.0482	+ 0.8588	+ 0.5927	+ 0.3881
$\vartheta_1^2 N_{110}^{(s)}$	+ 0.1348	+ 0.0436	— 0.0819	— 0.1703
$\vartheta_1^2 N_{101}^{(s)}$	— 0.0993	— 0.1192	— 0.1473	— 0.1689
$\vartheta_1^2 N_{011}^{(s)}$	— 0.0062	— 0.0417	— 0.0916	— 0.1330

TABLE X.

	1.0000	0.8422	0.6205
$\vartheta_1^3 B_{300}^{(s)}$	+ 0.6201	+ 0.0937	— 0.2701
$\vartheta_1^3 B_{210}^{(s)}$	+ 0.7865	+ 0.1988	— 0.1978
$\vartheta_1^3 B_{120}^{(s)}$	+ 0.2577	— 0.0104	— 0.1466
$\vartheta_1^3 B_{030}^{(s)}$	+ 0.4075	+ 0.1220	— 0.0726
$\vartheta_1^3 B_{021}^{(s)}$	+ 0.1923	+ 0.0378	— 0.0549
$\vartheta_1^3 B_{212}^{(s)}$	+ 0.2065	— 0.0014	— 0.1331
$\vartheta_1^3 B_{003}^{(s)}$	+ 0.2342	+ 0.0862	— 0.0227
$\vartheta_1^3 B_{102}^{(s)}$	— 0.1002	— 0.1622	— 0.1731
$\vartheta_1^3 B_{201}^{(s)}$	+ 0.2817	+ 0.0422	— 0.1110
$\vartheta_1^3 B_{111}^{(s)}$	+ 0.1394	+ 0.0437	+ 0.0310

TABLE XI.

	1.0000	0.8422	0.6205
$\mathfrak{P}_1^4 B_{400}^{(s)}$	+ 1.5776	+ 0.0004	— 0.2952
$\mathfrak{P}_1^4 B_{810}^{(s)}$	— 0.6030	— 0.3645	+ 0.0807
$\mathfrak{P}_1^4 B_{220}^{(s)}$	+ 1.8696	+ 0.1504	— 0.1690
$\mathfrak{P}_1^4 B_{130}^{(s)}$	— 0.1314	— 0.1493	+ 0.0247
$\mathfrak{P}_1^4 B_{400}^{(s)}$	+ 0.4692	+ 0.0583	— 0.0154
$\mathfrak{P}_1^4 B_{081}^{(s)}$	+ 0.0270	— 0.0894	— 0.0314
$\mathfrak{P}_1^4 B_{022}^{(s)}$	+ 1.0008	+ 0.1105	— 0.0607
$\mathfrak{P}_1^4 B_{013}^{(s)}$	+ 1.1181	+ 0.3232	+ 0.0560
$\mathfrak{P}_1^4 B_{004}^{(s)}$	+ 0.6483	+ 0.1671	— 0.0084
$\mathfrak{P}_1^4 B_{103}^{(s)}$	— 0.1591	— 0.1011	+ 0.0182
$\mathfrak{P}_1^4 B_{202}^{(s)}$	+ 1.6343	+ 0.1560	— 0.2381
$\mathfrak{P}_1^4 B_{301}^{(s)}$	— 1.1396	— 0.2920	+ 0.1245
$\mathfrak{P}_1^4 B_{211}^{(s)}$	+ 0.9084	+ 0.1518	+ 0.0201
$\mathfrak{P}_1^4 B_{121}^{(s)}$	+ 0.8277	+ 0.2351	+ 0.0517
$\mathfrak{P}_1^4 B_{112}^{(s)}$	— 0.3364	— 1.0339	+ 0.1797

23. Before discussing the tables we will derive the β -characteristics. They are

$$\begin{aligned} \sigma_1^{(s)} &= \sqrt{N_{200}^{(s)}}; & \sigma_2^{(s)} &= \sqrt{N_{020}^{(s)}}; & \sigma_3^{(s)} &= \sqrt{N_{002}^{(s)}}; \\ \gamma_{12}^{(s)} &= \frac{N_{110}^{(s)}}{\sigma_1^{(s)} \sigma_2^{(s)}}; & \gamma_{13}^{(s)} &= \frac{N_{101}^{(s)}}{\sigma_1^{(s)} \sigma_3^{(s)}}; & \gamma_{23}^{(s)} &= \frac{N_{011}^{(s)}}{\sigma_2^{(s)} \sigma_3^{(s)}}; \\ \tilde{\gamma}_{ijk}^{(s)} &= \frac{B_{ijk}^{(s)}}{\sigma_1^{(s)i} \sigma_2^{(s)j} \sigma_3^{(s)k}}. \end{aligned}$$

We see that, save the dispersions σ , *these characteristics are abstract numbers of dimension zero*. They are independent of $\mathfrak{P}_1$, and it will be shown later on *that they are practically independent also if $\mathfrak{P}_1$ varies with galactic latitude*.

We find the following tables of the β -characteristics for the different q' .

TABLE XII.

q'	1.0000	0.8422	0.6205	0.4500
$\mathfrak{P}_1 \sigma_1^{(s)}$	1.8412	1.6674	1.3870	1.1247
$\mathfrak{P}_1 \sigma_2^{(s)}$	1.2690	1.1308	0.9024	0.6602
$\mathfrak{P}_1 \sigma_3^{(s)}$	1.0238	0.9267	0.7699	0.6230
$\gamma_{12}^{(s)}$	+ 0.058	+ 0.023	— 0.066	— 0.241
$\gamma_{13}^{(s)}$	— 0.053	— 0.077	— 0.138	— 0.242
$\gamma_{32}^{(s)}$	— 0.005	— 0.040	— 0.134	— 0.332

TABLE XIII.

q'	1.0000	0.8422	0.6205	
$\beta_{300}^{(s)}$	+ 0.0993	+ 0.0202	— 0.1012	0.80
$\beta_{210}^{(s)}$	+ 0.1830	+ 0.0603	— 0.1139	0.76
$\beta_{120}^{(s)}$	+ 0.0869	— 0.0044	— 0.1298	0.85
$\beta_{030}^{(s)}$	+ 0.2000	+ 0.0734	— 0.0888	0.74
$\beta_{021}^{(s)}$	+ 0.1166	+ 0.0291	— 0.0876	0.80
$\beta_{012}^{(s)}$	+ 0.1553	— 0.0014	— 0.2468	0.85
$\beta_{003}^{(s)}$	+ 0.2162	+ 0.1083	— 0.0497	0.70
$\beta_{102}^{(s)}$	— 0.0519	— 0.1132	— 0.2106	
$\beta_{201}^{(s)}$	+ 0.0812	+ 0.0164	— 0.0749	0.80
$\beta_{111}^{(s)}$	+ 0.0583	+ 0.0239	+ 0.0322	

TABLE XIV.

q'	1.0000	0.8422	0.6205	
$\beta_{400}^{(s)}$	+ 0.1373	+ 0.0000	— 0.0797	0.84
$\beta_{310}^{(s)}$	— 0.0761	— 0.0663	+ 0.0335	0.66
$\beta_{220}^{(s)}$	+ 0.3423	+ 0.0385	— 0.1079	0.81
$\beta_{130}^{(s)}$	— 0.0349	— 0.0539	+ 0.0242	0.68
$\beta_{040}^{(s)}$	+ 0.1809	+ 0.0296	— 0.0239	0.78
$\beta_{031}^{(s)}$	+ 0.0129	— 0.0580	— 0.0554	0.91
$\beta_{022}^{(s)}$	+ 0.2818	+ 0.0463	— 0.0532	0.79
$\beta_{013}^{(s)}$	+ 0.8211	+ 0.3428	+ 0.1360	
$\beta_{004}^{(s)}$	+ 0.5901	+ 0.2265	— 0.0239	0.64
$\beta_{103}^{(s)}$	— 0.0805	— 0.0761	+ 0.0287	0.71
$\beta_{202}^{(s)}$	+ 0.9682	+ 0.1294	— 0.4932	0.81
$\beta_{301}^{(s)}$	— 0.1783	— 0.0679	+ 0.0606	0.73
$\beta_{211}^{(s)}$	+ 0.2063	+ 0.0497	+ 0.0150	
$\beta_{121}^{(s)}$	+ 0.2727	+ 0.1084	+ 0.0594	
$\beta_{112}^{(s)}$	— 0.1578	— 0.0200	+ 0.2422	0.82

The most remarkable feature of the characteristics is the systematically low values for q' of the order of magnitude 0.8. Evidently the correlation coefficients show nothing new, they are only significant of the variation of the vertex directions for different q' . But it is seen that *nearly all the characteristics of the third and the fourth order change sign for q' between 0.84 and 0.62*. The last column of tables XIII and XIV contains the value of q' for which the characteristic is zero. Some of the characteristics do not change sign for any plausible value of q' . For them the last columns are empty.

Of course nothing definite can be gathered from these figures regarding the true value of q' . There is nothing, to our knowledge, that forces us to take for q' a value that makes the higher characteristics small, *it is only a very suggestive fact that they will, on the whole, be small if we take a value of q' between 0.7 and 0.8*.*

Besides, other suggestive circumstances can also be found. Computing for instance by the formula (26) of § 6 the coordinates of the mode (the vector of maximum frequency) we find the following values:

* It is also to be noted that the skewness and excess of the radial velocities are found to be small.

q'	1.0000	0.8422	0.6205
X_1	+ 0.3329	— 0.0570	— 0.6440
Y_1	+ 0.9383	+ 0.2791	— 0.6591
Z_1	+ 0.8524	+ 0.3704	— 0.3116

Obviously, for the above values of q' , the characteristics are too great to allow the use of the formula (26). But one thing we may conclude, namely that for values of q' about 0.75 the mode will not lie far out of the X -axis. As by $q' = 0.75$ our galactic system of coordinates is nearly coincident with the vertex system we may say: for $q' = 0.75$ the mode will practically lie on the axis toward the principal vertex. Evidently this is required by the two stream-hypothesis.

As one more in a way suggestive fact I will remark that for q' somewhere near 0.75 the total excess E of the small velocity components as given in formula (32**) will disappear.

24. We have just pointed out that for q' about 0.75 the mode vector is directed as required by the two-stream hypothesis. We will here look a little further into the question what our figures have to say about this hypothesis. The two-stream hypothesis requires that the frequency function of stellar motion shall be the addition of two normal »components».

We denote by

n_1 and n_2 the relative number of stars in the two drifts,

α_1 and α_2 the dispersions of the velocities in the two drifts,

m_1 and m_2 the stream velocities relative to the mean velocity of both the drifts.

According to CHARLIER* we then have for the characteristics of the resultant velocity distribution referred to the vertex system (Compare also our equations (68), (68*), (68**) and (68***)):

$$\begin{aligned} n_1 + n_2 &= 1 \\ n_1 m_1 + n_2 m_2 &= 0 \end{aligned}$$

$$N_{200} = n_1 \alpha_1^2 + n_2 \alpha_2^2 + n_1 n_2 (m_1 - m_2)^2$$

$$N_{020} = n_1 \alpha_1^2 + n_2 \alpha_2^2$$

$$N_{002} = n_1 \alpha_1^2 + n_2 \alpha_2^2$$

$$N_{110} = 0 \quad N_{101} = 0 \quad N_{011} = 0.$$

$$B_{300} = -\frac{1}{2} n_1 n_2 (\alpha_1^2 - \alpha_2^2) (m_1 - m_2) + \frac{1}{6} n_1 n_2 (n_1 - n_2) (m_1 - m_2)^3$$

$$B_{120} = -\frac{1}{2} n_1 n_2 (\alpha_1^2 - \alpha_2^2) (m_1 - m_2)$$

$$B_{102} = -\frac{1}{2} n_1 n_2 (\alpha_1^2 - \alpha_2^2) (m_1 - m_2)$$

$$B_{210} = 0 \quad B_{201} = 0 \quad B_{021} = 0 \quad B_{030} = 0 \quad B_{012} = 0 \quad B_{003} = 0.$$

* CHARLIER l. c. pp. 94 and 95.

$$\begin{aligned}
B_{400} &= \frac{1}{8} n_1 n_2 (\alpha_1^2 - \alpha_2^2)^2 - \frac{1}{4} n_1 n_2 (n_1 - n_2) (\alpha_1^2 - \alpha_2^2) (m_1 - m_2)^2 + \\
&\quad + \frac{1}{24} n_1 n_2 (1 - 6n_1 n_2) (m_1 - m_2)^4 \\
B_{220} &= B_{202} = B_{022} = \frac{1}{4} n_1 n_2 (\alpha_1^2 - \alpha_2^2)^2 \\
B_{040} &= B_{004} = \frac{1}{8} n_1 n_2 (\alpha_1^2 - \alpha_2^2)^2 \\
B_{310} &= B_{301} = B_{130} = B_{103} = B_{031} = B_{013} = 0.
\end{aligned}$$

Now, firstly, as the ellipsoid is three axial the condition $N_{020} = N_{002}$ is not fulfilled.

For $q' = 0.75$, however, the conditions $B_{030} = B_{003} = B_{210} = B_{201} = 0$ are nearly fulfilled.

The conditions $B_{310} = B_{301} = B_{130} = B_{103} = B_{031} = B_{013} = 0$ cannot be said to be fulfilled, and B_{220} , B_{022} and B_{202} are not as is required positive. B_{004} is, however, positive.

Before going further we will take a look at the values found by EDDINGTON for the characteristics of the two drifts. From his new book »Stellar movements and the structure of the universe» I find that he has found

$$\alpha_1 = \alpha_2 \quad n_1 = 0.6 \quad n_2 = 0.4$$

and taking the square dispersion transverse to the relative stream direction as half the unity

$$n_1 \alpha_1^2 + n_2 \alpha_2^2 = 0.5 \quad m_1 - m_2 = -1.87.$$

Hence we obtain from our formulæ

$$\begin{aligned}
N_{200} &= 1.34 & N_{020} &= N_{002} = 0.5 \\
B_{300} &= -0.105 & B_{400} &= -0.0538
\end{aligned}$$

and all the other characteristics zero.

As the unities differ from ours we compute the β -characteristics which are independent of the unity. We find

$$\beta_{300} = -0.033 \quad \beta_{400} = -0.030$$

and the others zero.

For $q' = 0.75$ we obtain from our figures

$$\beta_{300} = -0.028 \quad \beta_{400} = -0.047$$

which is a very pretty agreement.

To my opinion the results cited from EDDINGTON principally constitute a determination of the skewness and the excess in the vertex direction*, and the above small calculation is, indeed, a determination of the constant q' being strongly in

* To avoid misapprehension I once more call attention to the fact that the results of EDDINGTON, being based on a study of the position angles of the proper motions do not necessarily represent the true distribution of the linear velocities. More strictly they may be said to characterize the distribution of movements in a constructed system of stars giving rise to the observed directions of motion on the sky and as far as then possible having the two-stream property. The results of EDDINGTON are thus »true» in a similar sense as our results if we adopt a value of q' only on account of it making the deviations from the normal distribution as small as possible.

support of the value 0.75. The definite bearing of our figures on the two-stream hypothesis is not to be asserted before we have computed the mean errors in the characteristics. We shall come back to the question in § 30.

25. By the discussions of this chapter it has been made probable that the constant q' has a value about 0.75. The evidence in favour of this value is: The vertices constitute a galactic system as has been found from the radial velocities; the transgalactic axis has the same value as deduced from the radial velocities; the coefficients of the skewness and the excess are small; the skewness and excess of the motions as projected on the principal axis of polarity is the same as found when using the results of EDDINGTON for the two-stream hypothesis. Accordingly we will adopt $q' = 0.75$ for the linear motions of the stars and give the characteristics for this value of q' . Before this, however, we will assert the effect of a variation of the mean parallax ϑ_1 with galactic latitude. In this connection we will also work out the methods to derive the mean errors of the characteristics.

CHAPTER VIII.

The mean errors and the effect of a varying β_1 on the positions of the vertices and the values of the characteristics.

24. We shall in this chapter at some length derive the method to find the mean errors of the characteristics. It is then necessary first to find the mean errors of the characteristics of the motion as projected on a square. This will be the care of this paragraph, which also may be regarded as a supplement to the mathematical theory of the correlation function of the A -type as developed in chapter I.

The problem to find the mean errors of the characteristics of a correlation surface has, though not yet published, been considered by CHARLIER. By his kindness I have had access to a manuscript in which he gives the general formulæ necessary for solving the problem, and the outlines of the treatment of the higher characteristics.

Denoting by $\varepsilon(x)$ the mean error of the quantity x , CHARLIER gives the following general formula for the mean errors of the moments of the correlation surface (N = the number of stars):

$$(69) \quad \varepsilon(N_{l, m}) = \frac{1}{\sqrt{N}} \sqrt{N_{2l, 2m} - N_{l, m}^2}.$$

From this formula it follows immediately that

$$(70) \quad \begin{aligned} \varepsilon(N_{20}) &= \frac{N_{20}}{\sqrt{N}} \sqrt{2 + 24\beta_{40}} \\ \varepsilon(N_{11}) &= \frac{\sqrt{N_{20} N_{02}}}{\sqrt{N}} \sqrt{1 + r^2 + 4\beta_{22}} \\ \varepsilon(N_{02}) &= \frac{N_{02}}{\sqrt{N}} \sqrt{2 + 24\beta_{04}}. \end{aligned}$$

In the following we will confine ourselves to *the case that the distribution is nearly normal*. We then have

$$\begin{aligned}
 \varepsilon(N_{20}) &= \sqrt{2} \frac{N_{20}}{\sqrt{N}} \\
 (71) \quad \varepsilon(N_{11}) &= \frac{1}{\sqrt{N}} \sqrt{N_{20} N_{02} + N_{11}^2} \\
 \varepsilon(N_{02}) &= \sqrt{2} \frac{N_{02}}{\sqrt{N}},
 \end{aligned}$$

and, furthermore, the moments $N_{l,m}$ and $N_{2l,2m}$ may be expressed in terms of N_{20} , N_{11} , N_{02} . As we wish to find the mean errors of the moments and characteristics to the fourth order we must needs find the expressions for the above moments to the eighth order. To the fourth order they are already given in formulæ (14) and (15). For the sixth and eighth orders I use the following method.*

Putting as in § 14

$$\delta_1 = \begin{vmatrix} A_1 & F_1 \\ F_1 & B_1 \end{vmatrix}$$

we know that

$$(72) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-1/2 (A_1 x^2 + B_1 y^2 + 2F_1 xy)} = \frac{1}{\delta_1^{1/2}}.$$

By differentiating this equation three and four times with regard to the parameters A_1 , B_1 , F_1 , and multiplying with $\delta_1^{1/2}$, all the moments of the sixth resp eighth order come forth. We obtain, taking note of equations (28*) and (29):

$$\begin{aligned}
 (73) \quad N_{60} &= 15N_{20}^3 \\
 N_{51} &= 15N_{20}^2 N_{11} \\
 N_{42} &= 3N_{20}^2 N_{02} + 12N_{20} N_{11}^2 \\
 N_{33} &= 9N_{20} N_{02} N_{11} + 6N_{11}^3 \\
 N_{24} &= 3N_{02}^2 N_{20} + 12N_{02} N_{11}^2 \\
 N_{15} &= 15N_{02}^2 N_{11} \\
 N_{06} &= 15N_{02}^3, \\
 \\
 (73^*) \quad N_{80} &= 105N_{20}^4 \\
 N_{62} &= 15N_{20}^3 N_{02} + 90N_{20}^2 N_{11}^2 \\
 N_{44} &= 9N_{20}^2 N_{02}^2 + 72N_{20} N_{02} N_{11}^2 + 24N_{11}^4 \\
 N_{26} &= 15N_{20} N_{02}^3 + 90N_{02}^2 N_{11}^2 \\
 N_{08} &= 105N_{02}^4,
 \end{aligned}$$

and the remaining moments of the eighth order, which are not needed here, are as easily derived.

Introducing these expressions into equation (69) and remembering the formulæ (14) and (15) we get

* Compare § 4.

$$\begin{aligned}
(74) \quad \varepsilon(N_{30}) &= \frac{1}{\sqrt{N}} \sqrt{15N_{20}^3} \\
\varepsilon(N_{21}) &= \frac{1}{\sqrt{N}} \sqrt{3N_{20}^2 N_{02} + 12N_{20} N_{11}^2} \\
\varepsilon(N_{12}) &= \frac{1}{\sqrt{N}} \sqrt{3N_{20} N_{02}^2 + 12N_{02} N_{11}^2} \\
\varepsilon(N_{03}) &= \frac{1}{\sqrt{N}} \sqrt{15N_{02}^3} \\
\varepsilon(N_{40}) &= \frac{1}{\sqrt{N}} \sqrt{96N_{20}^4} \\
\varepsilon(N_{31}) &= \frac{1}{\sqrt{N}} \sqrt{15N_{20}^3 N_{02} + 81N_{20}^2 N_{11}^2} \\
(74^*) \quad \varepsilon(N_{22}) &= \frac{1}{\sqrt{N}} \sqrt{8N_{20}^2 N_{02}^2 + 68N_{20} N_{02} N_{11}^2 + 20N_{11}^4} \\
\varepsilon(N_{13}) &= \frac{1}{\sqrt{N}} \sqrt{15N_{20} N_{02}^3 + 81N_{02}^2 N_{11}^2} \\
\varepsilon(N_{04}) &= \frac{1}{\sqrt{N}} \sqrt{96N_{02}^4}.
\end{aligned}$$

By the equations (16) we now have immediately

$$\begin{aligned}
(75) \quad \varepsilon(B_{30}) &= \frac{1}{6\sqrt{N}} \sqrt{15N_{20}^3} \\
\varepsilon(B_{21}) &= \frac{1}{2\sqrt{N}} \sqrt{3N_{20}^2 N_{02} + 12N_{20} N_{11}^2} \\
\varepsilon(B_{12}) &= \frac{1}{2\sqrt{N}} \sqrt{3N_{20} N_{02}^2 + 12N_{02} N_{11}^2} \\
\varepsilon(B_{03}) &= \frac{1}{6\sqrt{N}} \sqrt{15N_{02}^3}
\end{aligned}$$

by which the mean errors of the third characteristics of a nearly normal correlation surface are given. For the β -characteristics we have

$$\begin{aligned}
(76) \quad \varepsilon(\beta_{30}) &= \frac{\sqrt{15}}{6\sqrt{N}} = \varepsilon(\beta_{03}) \\
\varepsilon(\beta_{21}) &= \frac{1}{2\sqrt{N}} \sqrt{3 + 12r^2} = \varepsilon(\beta_{12}).
\end{aligned}$$

As the skewness in x is $S_x = 3\beta_{30}$ we have the well known formula

$$\varepsilon(S_x) = \varepsilon(S_y) = \frac{\sqrt{3.75}}{\sqrt{N}} = \frac{1.936492}{\sqrt{N}}.$$

The mean errors of the characteristics of the fourth order it is somewhat more complicate to derive as they are compound functions of the moments. The method proposed by CHARLIER is the following:

If by $\nu_{11}(x_1, x_2)$ we denote the mean of the quantity $x_1 x_2$ we have for the mean error of the function $h(x_1, x_2, x_3, \dots x_s)$

$$(77) \quad \varepsilon(h) = \sqrt{\sum_{i=1}^s \left(\frac{\partial h}{\partial x_i} \right)^2 \varepsilon(x_i)^2 + 2 \sum_{i,j=1}^s \frac{\partial h}{\partial x_i} \frac{\partial h}{\partial x_j} \nu_{11}(x_i, x_j)}.$$

Now by the fourth characteristics the moments of the fourth and second orders play the part of the quantities $x_1, x_2, \dots x_s$ in the above function h . Thus we get the mean errors of the fourth characteristics expressed in terms of the second moments, the mean errors of the fourth and second moments and the ν_{11} of these moments. These mean errors are taken from equations (71) and (74) and for the quantity ν_{11} between two moments CHARLIER gives the general formula

$$(78) \quad \nu_{11}(N_{pq}, N_{p'q'}) = \frac{1}{N} (N_{p+p'q+q'} - N_{pq} N_{p'q'}).$$

Now by the formula (17), (71), (74*), (77) and (78) we obtain

$$(75^*) \quad \begin{aligned} \varepsilon(B_{40}) &= \frac{1}{24\sqrt{N}} \sqrt{24N_{20}^4} \\ \varepsilon(B_{31}) &= \frac{1}{6\sqrt{N}} \sqrt{6N_{20}^3 N_{02} + 18N_{20}^2 N_{11}^2} \\ \varepsilon(B_{22}) &= \frac{1}{4\sqrt{N}} \sqrt{4N_{20}^2 N_{02}^2 + 16N_{20} N_{02} N_{11}^2 + 4N_{11}^4} \\ \varepsilon(B_{13}) &= \frac{1}{6\sqrt{N}} \sqrt{6N_{20} N_{02}^3 + 18N_{02}^2 N_{11}^2} \\ \varepsilon(B_{04}) &= \frac{1}{24\sqrt{N}} \sqrt{24N_{02}^4} \end{aligned}$$

and for the β -characteristics

$$(76^*) \quad \begin{aligned} \varepsilon(\beta_{40}) &= \frac{1}{\sqrt{24N}} = \varepsilon(\beta_{04}) & \varepsilon(\beta_{22}) &= \frac{1}{4\sqrt{N}} \sqrt{4 + 16r^2 + 4r^4} \\ \varepsilon(\beta_{31}) &= \frac{1}{6\sqrt{N}} \sqrt{6 + 18r^2} = \varepsilon(\beta_{13}) \end{aligned}$$

As the excess in x is $E_x = 3\beta_{40}$ and similarly for y we again come back to a well known formula viz:

$$\varepsilon(E_x) = \varepsilon(E_y) = \frac{\sqrt{0.375}}{\sqrt{N}} = \frac{0.612372}{\sqrt{N}}.$$

25. Now, to derive the mean errors of the characteristics of the motion in space computed for, say, the galactic system of coordinates we may proceed by a method similar to that generally in use for least squares solutions. Let us by $\xi_p^{(i)}$ denote a characteristic of the i th order referred to the galactic system. It is clear that by a least squares solution the values found for the characteristics $\xi_p^{(i)}$ are the same whether we apply the solution to equations of condition connecting the squares with the galactic system directly, or, as has actually been done, "first go by the system of the equator. Furthermore we should have found the same values of $\xi_p^{(i)}$ if the squares had been chosen equally symmetrical to the galactic plane as the actual squares to the equator. Then we should also have obtained normal equations for the $\xi_p^{(i)}$ having the same left membra as those of Chapter VI, as these membra depend only on the direction cosines of the squares.

We shall suppose that we have applied the least squares method to such a galactic system of squares, which, to avoid confusion, we will call the g -squares. Then between each g -square and the galactic system of coordinates we have equations of condition of the form:

$$(79) \quad \begin{aligned} \vartheta_1^i (a_{11}^{(i)} \xi_1^{(i)} + a_{12}^{(i)} \xi_2^{(i)} + a_{13}^{(i)} \xi_3^{(i)} + \dots + a_{1r}^{(i)} \xi_r^{(i)}) &= \gamma_{11}^{(i)} \\ \vartheta_1^i (a_{21}^{(i)} \xi_1^{(i)} + a_{22}^{(i)} \xi_2^{(i)} + a_{23}^{(i)} \xi_3^{(i)} + \dots + a_{2r}^{(i)} \xi_r^{(i)}) &= \gamma_{12}^{(i)} \\ \dots &\dots \\ \vartheta_1^i (a_{s1}^{(i)} \xi_1^{(i)} + a_{s2}^{(i)} \xi_2^{(i)} + a_{s3}^{(i)} \xi_3^{(i)} + \dots + a_{sr}^{(i)} \xi_r^{(i)}) &= \gamma_{1s}^{(i)}. \end{aligned}$$

Having adopted a value of q' , the $\gamma_q^{(i)}$ are the »observed» characteristics of the linear* motion as projected on the g -square. Naturally by the multinomial theorem we have

$$r = \frac{3 \cdot 4 \cdot 5 \dots (3 + i - 1)}{[i]} \quad s = \frac{2 \cdot 3 \cdot 4 \dots (2 + i - 1)}{[i]}.$$

On account of the choice of the g -squares the coefficients $a_{jk}^{(i)}$ are the same as the coefficients tabulated in tables II, III and IV so that:

$$(80) \quad \begin{aligned} a_{1k}^{(2)} &= \alpha_k; & a_{2k}^{(2)} &= \beta_k; & a_{3k}^{(2)} &= \gamma_k, \\ a_{1k}^{(3)} &= a_k; & a_{2k}^{(3)} &= b_k; & a_{3k}^{(3)} &= c_k; & a_{4k}^{(3)} &= d_k, \\ a_{1k}^{(4)} &= a'_k; & a_{2k}^{(4)} &= b'_k; & a_{3k}^{(4)} &= c'_k; & a_{4k}^{(4)} &= d'_k; & a_{5k}^{(4)} &= e'_k. \end{aligned}$$

The normal equations are now the same as in Chapter VI. A glance at them will show that, with an approximation sufficient for our purposes, we may write them

$$(81) \quad \vartheta_1^i [a_{1p} a_{1p}] \xi_p^{(i)} = [a_{1p} \gamma_{11}^{(i)}].$$

Here

$$[a_{1p}^2] = \Sigma (a_{1p}^2 + a_{2p}^2 + a_{3p}^2 + a_{4p}^2), \quad [a_{1p} \gamma_{11}^{(i)}] = \Sigma (a_{1p} \gamma_{11}^{(i)} + a_{2p} \gamma_{12}^{(i)} \dots a_{sp} \gamma_{1s}^{(i)}),$$

the sum being taken over all the squares**.

* By the linear motion of a square we then mean the angular motion of a star placed at a distance corresponding to the mean parallax of the square.

** The index (i) of the a_{jp} is here omitted for the sake of convenience.

According to (77) the sought mean errors of the characteristics will be

$$(82) \quad [a_{1p} a_{1p}] \varepsilon (\vartheta_1^{(i)} \xi_p^{(i)})^2 = \frac{[a_{1p}^2 \varepsilon (\eta_1^{(i)})^2] + 2 \sum a_{qp} a_{q1p} \gamma_{11} (\gamma_q^{(i)}, \gamma_{q1}^{(i)})}{[a_{1p} a_{1p}]}.$$

In the method of least squares the »observations» $\gamma_q^{(i)}$ are generally supposed to be independent of each other so that the last term in the right membrum disappears. Furthermore, if weights of the form

$$w_q^{(i)} = \frac{\mu^{(i)2}}{\varepsilon (\gamma_q^{(i)})^2}$$

had been applied to the equations (79) the formula (82) would take the familiar form

$$(82^*) \quad [w a_{1p} a_{1p}] \varepsilon (\vartheta_1^i \xi_p^{(i)})^2 = \mu^{(i)2}.$$

Now, the mathematical treatment of the questions connected with finding the mean errors is brought to an end, a treatment, which for the sake of completeness I have considered appropriate to carry out at some length. Evidently the numerical determinations may be taken more in a way to obtain only the order of magnitude. The question is then only: which are the values to be assigned to the quantities $\mu^{(i)}$, which are a sort of general mean errors of the characteristics of the i th order in a g -square?

Denoting by γ_{ij} the direction cosines of a g -square we have for the second moments in the square the equations

$$(83) \quad \begin{aligned} N_{20} &= \gamma_{11}^2 N_{200}^{(s)} + \gamma_{21}^2 N_{020}^{(s)} \\ N_{02} &= \gamma_{12}^2 N_{200}^{(s)} + \gamma_{22}^2 N_{020}^{(s)} + \gamma_{32}^2 N_{002}^{(s)} \\ N_{11} &= \gamma_{11} \gamma_{12} (N_{200}^{(s)} - N_{020}^{(s)}), \end{aligned}$$

and denoting by β and λ galactic latitude and longitude we have

$$\gamma_{11} = -\sin \lambda, \quad \gamma_{21} = \cos \lambda, \quad \gamma_{12} = -\sin \beta \cos \lambda, \quad \gamma_{22} = -\sin \beta \sin \lambda, \quad \gamma_{32} = \cos \beta.$$

Putting $q' = 0.75$ we see from the following chapter that

$$\vartheta_1^2 N_{200}^{(s)} = 2.4243 \quad \vartheta_1^2 N_{020}^{(s)} = 1.0860 \quad \vartheta_1^2 N_{002}^{(s)} = 0.7482.$$

Now $\gamma_{11} \gamma_{12}$ is the coefficient γ_1 of table II, which we see is generally small, never exceeding the value 0.3368.

Hence N_{11} is always smaller than 0.45 and generally much smaller. Compared to N_{20} and N_{02} we will neglect the effect of N_{11} . For N_{20} and N_{02} we will take their mean values which are obtained by multiplying the equations (83) with $\cos \beta d\lambda d\beta$ and integrating over the whole sphere.

Thus we find

$$\begin{aligned} \vartheta_1^2 M(N_{20}) &= {}^{1/2} \vartheta_1^2 (N_{200}^{(s)} + N_{020}^{(s)}) = 1.75 \\ \vartheta_1^2 M(N_{02}) &= {}^{1/6} \vartheta_1^2 (N_{200}^{(s)} + N_{020}^{(s)} + 4 N_{002}^{(s)}) = 1.08 \end{aligned}$$

and the mean number of stars in a square

$$N = \frac{4041}{24} = 169.$$

As by $q' = 0.75$ the distribution is not far from normal we may use the equations (71), (75) and (75*), and obtain for the order of magnitude of the mean error in a g -square

$$\begin{aligned} \vartheta_1^2 \varepsilon(N_{20}) &= 0.19; \quad \vartheta_1^2 \varepsilon(N_{11}) = 0.11; \quad \vartheta_1^2 \varepsilon(N_{02}) = 0.12, \\ \vartheta_1^3 \varepsilon(B_{30}) &= 0.11; \quad \vartheta_1^3 \varepsilon(B_{21}) = 0.12; \quad \vartheta_1^3 \varepsilon(B_{12}) = 0.10; \quad \vartheta_1^3 \varepsilon(B_{03}) = 0.06, \\ \vartheta_1^4 \varepsilon(B_{40}) &= 0.05; \quad \vartheta_1^4 \varepsilon(B_{31}) = 0.07; \quad \vartheta_1^4 \varepsilon(B_{22}) = 0.07; \quad \vartheta_1^4 \varepsilon(B_{13}) = 0.05; \quad \vartheta_1^4 \varepsilon(B_{04}) = 0.02. \end{aligned}$$

As adequate values we now take

$$\mu^{(2)} = 0.15$$

$$\mu^{(3)} = 0.10$$

$$\mu^{(4)} = 0.05$$

and have from (82*)

$$\varepsilon(\xi_p^{(i)} \vartheta_1^i) = \frac{\mu^{(i)}}{\sqrt{[a_{1p} a_{1p}]}} ,$$

where for $[a_{1p} a_{1p}]$ we have to take the coefficient of the characteristic $\xi_p^{(i)}$ in the normal equations of Chapter VI.

26. Of far greater importance than the mean errors are the *systematic* errors that may come forth by a systematic variation of the parameters q' and ϑ_1 in the several squares. As to the parameter q' we have at present no means to ascertain its variation. By ϑ_1 the matter is more ripe and we shall in the next paragraph try to determine its probable *variation with galactic latitude*. First we shall, however, deduce the formulæ of correction to be applied after having found the amount of the variation. This we are in a position to do by the equations of the preceding paragraph.

We denote by $\vartheta_1(\beta)$ the value of ϑ_1 in regions of the galactic latitude β . The g -squares now form rings of equal galactic latitude, in which $\vartheta_1(\beta)$ is assumed to be constant. By index β denoting that only the ring of squares having the latitude β is used we obtain from each ring normal equations of the approximate form

$$\vartheta_{1(\beta)}^i [a_{1p}^{(i)} a_{1p}^{(i)}]_{\beta} \xi_p^{(i)} = [a_{1p}^{(i)} \eta_1^{(i)}]_{\beta} .$$

Taking the sum for all the rings we obtain normal equations of the form

$$\xi_p^{(i)} \sum \vartheta_{1(\beta)}^i [a_{1p}^{(i)} a_{1p}^{(i)}]_{\beta} = [a_{1p}^{(i)} \eta_1^{(i)}] .$$

Putting

$$(84) \quad \sum \vartheta_{1(\beta)}^i [a_{1p}^{(i)} a_{1p}^{(i)}]_{\beta} = (\vartheta_1^i)_p [a_{1p}^{(i)} a_{1p}^{(i)}]$$

we come back to our old form

$$[a_{1p}^{(i)} a_{1p}^{(i)}] (\vartheta_1^i)_p \xi_p^{(i)} = [a_{1p}^{(i)} \eta_1^{(i)}] ,$$

save for the circumstance that we here have the »mean» values $(\vartheta_1^i)_p$ instead of ϑ_1 . By ϑ_1 denoting the mean parallax of the whole heavens as deduced by comparing

the »angular» velocity of the sun $\vartheta_1 S$ with the linear velocity found from the radial motions, we put

$$(84^*) \quad (H^i)_p = \frac{(\vartheta_1^i)_p}{\vartheta_1^i}.$$

Accordingly we correct for the variation of ϑ_1 with galactic latitude by multiplying our values found for $\vartheta_1^i \xi_p^{(i)}$ by the factor $\frac{1}{(H^i)_p}$.

The values thus corrected will be the true values of $\vartheta_1^i \xi_p^{(i)}$ in as much as they before were faulty on account of the variation.

Now we may at once ascertain the effect of the variation on the vertices. If the vertices coincide with the galactic system of coordinates, which is nearly the case, the moments $N_{110}^{(s)}$, $N_{101}^{(s)}$ and $N_{011}^{(s)}$ are all zero. The factorial correction cannot change them and accordingly the vertices will not be displaced. The same may be shown to be the case if the two axes of the ellipsoid lie in the plane of the Milky Way without necessarily coinciding with our specially chosen galactic system of coordinates. Shortly we may state that *if one axis points toward or in the vicinity of the pole of the Milky Way the vertices will be unaffected or nearly unaffected by the variation of ϑ_1 with galactic latitude.*

It will be of interest to see how the vertices are affected when not forming a galactic system, as is the case for low values of q' . The principal vertex, as is seen from table VII and fig. 1, will never move out of the Milky Way, since when q' decreases it moves nearly along that plane. It will therefore be very little changed by a correction for a varying ϑ_1 . Now we have found the shortest axis most near the galactic pole. For such values of q' as make its polar distance sensible the correction for a ϑ_1 growing with galactic latitude will increase that distance, so that the effect will be a rotation of the ellipsoid about the principal axis in a retrograde direction. A variation in ϑ_1 , great enough to exchange the ratio of the two smaller axes, will produce the effect that the *medium* axis comes up in the vicinity of the pole. Indeed, it seems, as I have found by a rough computation, as if even for $q' = 0.6$ a variation of ϑ_1 , great enough to make the ratio of the axes the same as found from the radial velocities, would make the medium axis point very near to the galactic pole, a fact that I consider to be of great interest.

The facts above stated are found by reasoning in the following way. Taking a galactic system of coordinates having its X-axis directed toward the principal vertex the polar distance φ of the transgalactic axis is given by the formula

$$\operatorname{tg} 2\varphi = \frac{2N_{011}}{N_{020} - N_{002}}, \quad (\text{uncorrected } N_{020} > N_{002})$$

and as the correction for a ϑ_1 growing with galactic latitude evidently decreases and finally even changes the sign of the difference $N_{020} - N_{002}$ the above statements are easily verified.

As regards the effect on the *Apex* it is easily concluded that the correction will tend to increase the transgalactic component of solar motion relative to the components in the Galaxy. Thus the Apex is found too near the plane of the Milky Way when the variation is not taken into account, a fact that has recently been pointed out by EDDINGTON.

By the actual determination of the correction factors $1/(H^i)_p$ we will assume that ϑ_1 has two values, one value in the Galaxy and another out of the Galaxy. The ratio of the latter to the former we denote by H . Further we assume the galactic ϑ_1 to belong to a belt 30° wide on either side of the plane of the Milky Way, that is the ring of the C — g -squares. The other value we accordingly have in the regions of less than 60° polar distance, that is the A - and B — g -squares.

Denoting the sum $[a_{1p}^{(i)} a_{1p}^{(i)}]$ by $S^{(i)}$ we have by (84) the equation

$$(\vartheta_1^i)_p = \frac{S_{AB}^{(i)} \cdot H^i + S_C^{(i)}}{S^{(i)}} \vartheta_1^i(C).$$

Putting

$$\frac{\vartheta_1(C)}{\vartheta_1} = k \quad \frac{S_{AB}^{(i)}}{S^{(i)}} = s_{AB}^{(i)} \quad \frac{S_C^{(i)}}{S^{(i)}} = s_C^{(i)}$$

we obtain

$$(85) \quad (H^i)_p = (s_{AB}^{(i)} H^i + s_C^{(i)}) k^i.$$

TABLE XV.

	$s_{AB}^{(2)}$	$s_C^{(2)}$		$s_{AB}^{(3)}$	$s_C^{(3)}$		$s_{AB}^{(4)}$	$s_C^{(4)}$
$N_{200}^{(s)}$	0.61	0.39	$B_{300}^{(s)}$	0.56	0.44	$B_{400}^{(s)}$	0.74	0.26
$N_{110}^{(s)}$	0.63	0.37	$B_{210}^{(s)}$	0.52	0.48	$B_{310}^{(s)}$	0.88	0.12
$N_{020}^{(s)}$	0.61	0.39	$B_{120}^{(s)}$	0.52	0.48	$B_{220}^{(s)}$	0.90	0.10
			$B_{030}^{(s)}$	0.56	0.44	$B_{130}^{(s)}$	0.88	0.12
$N_{011}^{(s)}$	0.52	0.48				$B_{040}^{(s)}$	0.74	0.26
$N_{101}^{(s)}$	0.52	0.48	$B_{201}^{(s)}$	0.43	0.57			
			$B_{021}^{(s)}$	0.43	0.57	$B_{301}^{(s)}$	0.50	0.50
$N_{002}^{(s)}$	0.19	0.81	$B_{111}^{(s)}$	0.37	0.63	$B_{031}^{(s)}$	0.50	0.50
						$B_{211}^{(s)}$	0.63	0.37
			$B_{102}^{(s)}$	0.25	0.75	$B_{121}^{(s)}$	0.63	0.37
			$B_{012}^{(s)}$	0.25	0.75			
			$B_{003}^{(s)}$	0.11	0.89	$B_{202}^{(s)}$	0.29	0.71
						$B_{022}^{(s)}$	0.29	0.71
						$B_{112}^{(s)}$	0.27	0.73
						$B_{103}^{(s)}$	0.15	0.85
						$B_{013}^{(s)}$	0.15	0.85
						$B_{004}^{(s)}$	0.06	0.94

The quantities $S^{(i)}$ we have already computed in the course of the preparation of the normal equations. Hence the coefficients $s^{(i)}$ are easily found. They are tabulated in table XV, in which the entries are the characteristics to be corrected by equation (85).

27. In order to gain an opinion of the amount of variation in ϑ_1 we will make a comparison between the mean velocity components x_0 and y_0 in the actual squares and the components of the solar motion $\vartheta_1 S$ found from the whole heavens.

The coordinates of a square being α and δ and of apex being A and D we have for the components of $\vartheta_1 S$ in the square

$$\begin{aligned} x_0' &= -\vartheta_1 S \cos D \cdot \sin(A - \alpha) \\ y_0' &= -\vartheta_1 S (\sin D \cos \delta - \sin \delta \cos D \cos(A - \alpha)). \end{aligned}$$

As $A = 273^\circ$ $D = 32^\circ$ we may write

$$\begin{aligned} x_0' &= \vartheta_1 S \gamma_{21} \cos 32^\circ \\ y_0' &= \vartheta_1 S (\gamma_{22} \cos 32^\circ - \gamma_{32} \sin 32^\circ). \end{aligned}$$

On page 45 we have found the value $\vartheta_1 S = 1.0275$. Using this value we find x_0' and y_0' as given in the first and third columns of table XVI. The second and fourth columns contain the observed values x_0 and y_0 . Taking the velocity of the sun $S = 19.4$ km we found

$$\vartheta_1 = 0.052$$

and using that value we have in the fifth and sixth columns tabulated the quantities

$$\vartheta_1(x) = \frac{x_0'}{x_0} \vartheta_1 \qquad \vartheta_1(y) = \frac{y_0'}{y_0} \vartheta_1$$

TABLE XVI.

	x_0'	x_0	y_0'	y_0	$\vartheta_1(x)$	$\vartheta_1(y)$
A_1	0.000	+ 0.180	— 0.936	— 0.625	—	0.034
A_2	0.000	+ 0.047	+ 0.766	+ 0.871	—	0.059
B_1	+ 0.831	+ 0.697	— 0.577	— 0.573	0.043	0.051
B_2	+ 0.514	+ 0.509	— 0.887	— 0.745	0.051	0.044
B_3	0.000	+ 0.161	— 1.004	— 0.969	—	0.050
B_4	— 0.514	— 0.492	— 0.887	— 0.671	0.050	0.040
B_5	— 0.831	— 0.992	— 0.577	— 0.638	0.062	0.057
B_6	— 0.831	— 0.707	— 0.194	— 0.245	0.044	0.065
B_7	— 0.514	— 0.744	+ 0.115	+ 0.225	0.075	(0.101)
B_8	0.000	+ 0.180	+ 0.214	+ 0.298	—	0.066
B_9	+ 0.514	+ 0.496	+ 0.115	— 0.019	0.050	—
B_{10}	+ 0.831	+ 0.658	— 0.194	— 0.153	0.041	0.041
C_1	+ 0.844	+ 0.744	— 0.585	— 0.663	0.046	0.059
C_2	+ 0.618	+ 0.781	— 0.683	— 0.775	0.066	0.059
C_3	+ 0.226	+ 0.492	— 0.740	— 0.708	(0.112)	0.050
C_4	— 0.226	— 0.249	— 0.740	— 0.698	0.057	0.049
C_5	— 0.618	— 0.752	— 0.683	— 0.961	0.063	0.073
C_6	— 0.844	— 0.838	— 0.585	— 0.632	0.051	0.056
C_7	— 0.844	— 0.789	— 0.472	— 0.403	0.043	0.044
C_8	— 0.618	— 0.834	— 0.374	— 0.576	0.070	0.080
C_9	— 0.226	— 0.249	— 0.313	— 0.322	0.057	0.052
C_{10}	+ 0.226	+ 0.320	— 0.318	— 0.198	0.073	0.032
C_{11}	+ 0.618	+ 0.622	— 0.374	— 0.474	0.052	0.066
C_{12}	+ 0.844	+ 1.033	— 0.472	— 0.164	0.063	(0.018)

Now, of course, the value of ϑ_1 does not vary so much from square to square as the quantities $\vartheta_1(x)$ and $\vartheta_1(y)$, which is clearly seen from the fact that $\vartheta_1(x)$ and $\vartheta_1(y)$ are generally not more equal to each other in the same square than in different squares. Most of the variation is to be attributed to the mean errors in x_0 and y_0 . These are

$$\varepsilon(x_0) = \frac{\sqrt{\nu_{20}}}{\sqrt{N}} \quad \varepsilon(y_0) = \frac{\sqrt{\nu_{02}}}{\sqrt{N}}$$

and on the whole they lie between the limits 0.15 and 0.05. As seen from the table the differences $x_0' - x_0$, and $y_0' - y_0$ generally are of the same order of magnitude as the mean errors, and consequently no weight can be attached to the values of $\vartheta_1(x)$ and $\vartheta_1(y)$ of the individual squares. The question is, however, not so hopeless when we seek for *systematical* variations.

Taking together the squares with respect to their galactic position we find:

Squares within 30° of the Galaxy

$$\begin{array}{l} C_{10}, B_9, B_{10}, B_1, B_2, B_8, C_4 \text{ (} C_3 \text{ excluded)} \\ \text{Mean } \vartheta_1(x) = 0.052^* \\ \text{Mean } \vartheta_1(y) = 0.045 \end{array} \left\} \text{Mean } \vartheta_1 = 0.049$$

Squares partly within 30° of the Galaxy

$$\begin{array}{l} C_9, B_8, A_1, A_2, C_{11} \\ \text{Mean } \vartheta_1(x) = 0.055 \\ \text{Mean } \vartheta_1(y) = 0.055 \end{array} \left\} \text{Mean } \vartheta_1 = 0.055$$

Squares without 30° of the Galaxy

$$\begin{array}{l} C_6, C_7, C_8, B_5, B_6, C_{12}, C_1, C_2, B_4, C_5 \text{ (} B_7 \text{ excluded).} \\ \text{Mean } \vartheta_1(x) = 0.058 \\ \text{Mean } \vartheta_1(y) = 0.059 \end{array} \left\} \text{Mean } \vartheta_1 = 0.059$$

From these figures we see that the probable variation is of a magnitude to make $H = \frac{0.059}{0.049} = 1.2$. The fact that the variation comes out independently from the x_0 and y_0 components strengthens the probability of this value, which in it self is very plausible.

For the value of k we obtain

$$k = \frac{0.049}{0.052} = 0.94.$$

For the factor of correction for the second characteristics we now obtain from (85)

$$(H^2)_p = (s_{A, B}^{(2)} (1.2)^2 + s_C^{(2)}) (0.94)^2,$$

which gives, using table XV,

* On account of our having neglected 3° in the right ascension of the Apex this value is slightly too great.

$$\begin{array}{lll} (H^2)_{200} = 1.12 & (H^2)_{110} = 1.13 & (H^2)_{020} = 1.12 \\ (H^2)_{011} = 1.08 & (H^2)_{101} = 1.08 & (H^2)_{002} = 0.95 \end{array}$$

Now each of the second characteristics $\vartheta_1^{-2} N_{ijk}^{(s)}$ shall be multiplied by the factor $\frac{1}{(H^2)_{ijk}}$ to be corrected for the probable variation of ϑ_1 .

It will be seen that the ratio of the shortest axis to the two galactic axes will be greatedened by a factor equal to $\sqrt{1.17}$. As the ratio of the two smaller axes amounts to about 0.83 we will have a corrected ratio of about 0.90. It will be of interest to see what may be the mean error of this ratio. We have approximately

$$\varepsilon^2 \left(\frac{N_{002}^{(s)}}{N_{020}^{(s)}} \right) = \left(\frac{1}{N_{020}^{(s)}} \right)^2 \varepsilon^2(N_{002}^{(s)}) + \left(\frac{N_{002}^{(s)}}{N_{020}^{(s)}} \right)^2 \varepsilon^2(N_{020}^{(s)}),$$

and as we shall in the next chapter find that

$$\vartheta_1^{-2} \varepsilon(N_{020}^{(s)}) = 0.04 = \vartheta_1^{-2} \varepsilon(N_{002}^{(s)}),$$

we see that

$$\varepsilon \left(\frac{N_{002}^{(s)}}{N_{020}^{(s)}} \right) = 0.056.$$

Calling the ratio of the two smaller axes for ρ we find

$$\rho = 0.90 \pm 0.028$$

Accordingly we conclude that even after correcting for the probable value of the variation of ϑ_1 with galactic latitude *we cannot regard the ellipsoid as an ellipsoid of revolution, though the excentricity in sections transverse to the major axis is small.*

A question of some interest in this connection is, what is the value of H required to make the ellipsoid of the same form as found from the radial velocities? Now the answer to this question depends on what value we adopt for q' . It is true that a unique solution giving H and q' is possible to obtain, but the solution of q' is very unsharp. The points will be seen clearly in the following treatment. We have from § 21, dividing by ϑ_1^{-2}

$$\begin{array}{l} (H^2)_{200} N_{200}^{(s)} = 1380.6 q' - 169.0 \\ (H^2)_{020} N_{020}^{(s)} = 749.6 q' - 174.1 \\ (H^2)_{002} N_{002}^{(s)} = 429.0 q' - 54.3 \end{array}$$

As by a pure galactic variation of ϑ_1 we have $(H^2)_{200} = (H^2)_{020}$ we might solve for the three unknowns $(H^2)_{200}$, $(H^2)_{002}$ and q' , inserting for the $N_{ijk}^{(s)}$ the values found by GYLLENBERG from the radial velocities. Referred to a system of coordinates very close to our system III GYLLENBERG has found values that, when means are taken for the spectral types *A*, *F* and *G*, give

$$N_{200} = 449.4 \quad N_{020} = 176.9 \quad N_{002} = 306.3$$

We now find

$$\begin{aligned}
 (86) \quad x &= (H^2)_{200} = 3.07 q' - 0.37 \\
 x &= (H^2)_{020} = 4.24 q' - 0.98 \\
 y &= (H^2)_{002} = 1.40 q' - 0.18
 \end{aligned}$$

The rigorous solution is

$$q' = 0.52 \quad x = 1.23 \quad y = 0.36$$

which gives especially an absurdly low value of $y = (H^2)_{002}$. From equations (85) we now find for H and k the equations

$$\begin{aligned}
 1.23 &= k^2(0.61 H^2 + 0.39) \\
 0.36 &= k^2(0.19 H^2 + 0.81),
 \end{aligned}$$

which give an imaginary value of H . This is in itself nothing surprising, it only means that with such values of x and y as above it is an absurdity to assume one value of ϑ_1 to be common for galactic latitude less than 30° and another value to be common for the rest of the sky. But as before said the rigorous solution of the equations (86) is illusory. Taking $q' = 0.75$ we find

$$(H^2)_{200} = 1.93 \quad (H^2)_{020} = 2.10 \quad (H^2)_{002} = 0.87$$

by which it is seen that here also we may regard $(H^2)_{200}$ and $(H^2)_{020}$ as equal, a thing that accentuates the fact that a solution for q' here is quite unsharp, even if at any price we should require the ellipsoids of the cross-motion and radial motion to coincide. Evidently the assumption on which the equation (85) is based is still insufficient with the above values of the $(H^2)_{ijk}$, but we may conclude that *in order to make the dispersions of stellar motion along the axes of system III compatible with the values found from the radial velocities we should have to assume that the mean parallax of the stars brighter than 6.0 in the galactic polar regions is about double the mean parallax of the stars in the Galaxy.*

Evidently the assumption of a varying ϑ_1 does not suffice to explain the disagreement between the two ellipsoids.

As regards the higher characteristics, it is principally the β -characteristics that interest us. We remind of the formula

$$\beta_{ijk}^{(s)} = \frac{B_{ijk}^{(s)}}{V \sqrt{N_{200}^{(s)} N_{020}^{(s)} N_{002}^{(s)}}}$$

and see by the equation (85) and the table XV that *they will be nearly independent of the variation in ϑ_1* at least when H is as small as 1.2. Accordingly we will not trouble for their corrections; it will suffice to hold in mind that the coefficients of skewness are found slightly too small and the coefficients of excess slightly too large.

CHAPTER IX.

Characteristics of distribution for the linear motions.

28. By the preceding investigations we have found that the parameter q' probably is of the order of magnitude 0.75. Adopting this value for q' we find from the results of § 18 for the axes of the velocity ellipsoid and its vertices the following values

$$\begin{aligned}\vartheta_1 \sigma_1' &= 1.5606 \pm 0.0140 \text{ Vertex I } \alpha = 273^{\circ}.9 \quad \delta = -18^{\circ}.2 \\ \vartheta_1 \sigma_2' &= 1.0432 \pm 0.0209 \text{ Vertex II } \alpha = 338^{\circ}.8 \quad \delta = +53^{\circ}.6 \\ \vartheta_1 \sigma_3' &= 0.8580 \pm 0.0242 \text{ Vertex III } \alpha = 194^{\circ}.8 \quad \delta = +31^{\circ}.0.\end{aligned}$$

The mean errors are computed as described in § 25 and the unit is here the class-breadth $\omega = 0''.05$. Wishing to have the axes expressed in km. per sec. we have to put $\vartheta_1 = 0.0529$. Thus we find

$$\begin{aligned}\sigma_1' &= 29.50 \pm 0.26 \text{ km. per sec.} \\ \sigma_2' &= 19.72 \pm 0.40 \text{ km. per sec.} \\ \sigma_3' &= 16.22 \pm 0.46 \text{ km. per sec.}\end{aligned}$$

These values are, however, as yet uncorrected for the probable variation of the mean parallax with galactic latitude. We have in the preceding paragraph found that variation to be of a magnitude to make the ratio of ϑ_1 in the polar regions to ϑ_1 in the galactic regions about 1.2. To correct the axes for the effect of such a variation we have found that the above values shall be divided by respectively $\sqrt{1.12}$, $\sqrt{1.12}$ and $\sqrt{0.95}$. Hence we obtain as the best values to be found for the axes

$$\begin{aligned}\sigma_1' &= 27.83 \pm 0.24 \text{ km. per sec.} \\ \sigma_2' &= 18.60 \pm 0.38 \text{ km. per sec.} \\ \sigma_3' &= 16.73 \pm 0.47 \text{ km. per sec.}\end{aligned}$$

For the sake of completeness we remark that the mean error in the above value of ϑ_1 is about ± 0.0009 , which corresponds to a simultaneous oscillation in the length of the axes in the same direction of about 2 %.

To find the mean errors in the positions of the vertices we take recourse to the equations (61) and (63). Suppose we have computed the second moments referred to a system of coordinates whose axes deviate by infinitesimal angles from the axes of the ellipsoid. Then the moments N_{110} , N_{101} and N_{011} would be small quantities of the first order. The roots s_1 , s_2 , s_3 of the cubic (61) would then be found to deviate from the moments N_{200} , N_{020} and N_{002} by small quantities of the second order. In equations (63) we then may write

$$\begin{aligned} N_{200} - s_1 &= 0; & N_{020} - s_1 &= N_{020} - N_{200}; & N_{002} - s_1 &= N_{002} - N_{200} \\ N_{200} - s_2 &= N_{200} - N_{020}; & N_{020} - s_2 &= 0; & N_{002} - s_2 &= N_{002} - N_{020} \\ N_{200} - s_3 &= N_{200} - N_{002}; & N_{020} - s_3 &= N_{020} - N_{002}; & N_{002} - s_3 &= 0. \end{aligned}$$

Denoting by λ and β the longitude and latitude of the vertices in the system of coordinates considered, we find, neglecting quantities of the second order,

For vertex I	For vertex II
$\operatorname{tg} \lambda_1 = \lambda_1 = \frac{N_{110}}{N_{200} - N_{020}}$	$\operatorname{tg} (90 - \lambda_2) = \frac{\pi}{2} - \lambda_2 = -\frac{N_{110}}{N_{200} - N_{020}}$
$\operatorname{tg} \beta_1 = \beta_1 = \frac{N_{101}}{N_{200} - N_{002}}$	$\operatorname{tg} \beta_2 = \beta_2 = \frac{N_{011}}{N_{020} - N_{002}}$

For vertex III

$$\begin{aligned} \operatorname{tg} (90 - \beta_3) \cos \lambda_3 &= -\frac{N_{101}}{N_{200} - N_{002}} = \varphi_3 \\ \operatorname{tg} (90 - \beta_3) \sin \lambda_3 &= -\frac{N_{011}}{N_{020} - N_{002}} = \psi_3 \end{aligned}$$

where φ_3 and ψ_3 are the longitudes of vertex III counted from the W -axis along the UW - and VW -planes.

Neglecting mean errors multiplied with the small quantities N_{110} , N_{101} and N_{011} we finally have

$$\begin{aligned} \varepsilon(\lambda_1) = \varepsilon(\lambda_2) &= \frac{\varepsilon(N_{110})}{N_{200} - N_{020}}; & \varepsilon(\beta_1) = \varepsilon(\varphi_3) &= \frac{\varepsilon(N_{101})}{N_{200} - N_{002}}; \\ \varepsilon(\beta_2) = \varepsilon(\psi_3) &= \frac{\varepsilon(N_{011})}{N_{020} - N_{002}}. \end{aligned}$$

We now insert the values given in the next paragraph for the second moments and their mean errors referred to system III and thus obtain

$$\varepsilon(\lambda_1) = \varepsilon(\lambda_2) = 1^{\circ}.7; \quad \varepsilon(\beta_1) = \varepsilon(\varphi_3) = 1^{\circ}.4; \quad \varepsilon(\beta_2) = \varepsilon(\psi_3) = 6^{\circ}.8.$$

Obviously we may now put it in the following way: The vertices lie as far as the mean errors are concerned within small ellipses of the following descriptions.

Vertex I. Within an ellipse having its major axis parallel to the plane of the Milky Way. The axes are $a = 1^{\circ}.7$ $b = 1^{\circ}.4$.

Vertex II. Within an ellipse having its major axis perpendicular to the plane of the Milky Way. Here the axes are $a = 6^{\circ}.8$ $b = 1^{\circ}.7$.

Vertex III. Within an ellips having its major axis perpendicular to the direction toward the vertex I. The axes are $a = 6^{\circ}.8$ $b = 1^{\circ}.4$.

In fig. 1 the vertices are denoted by a cross. The length of the limbs of the crosses denote the axes of the above error ellipses. Obviously the system of vertices may be regarded as a galactic system. Hence the systematic variation of ϑ_1 has no effect on their position.



Fig. 1.

The vertices for $q' = 1, 0.75, 0.62, 0.45$.

The dotted great circle is the plane of the Milky Way and the star is its pole.

29. The higher characteristics it is most expedient for us to compute referred to the system III. The difference between the characteristics referred to system III and those referred to the vertex-system *will be of the order of magnitude of the mean errors*. Thus the figures given in this paragraph might as well be regarded as belonging to the vertex-system of coordinates.

I here give all the characteristics referred to system III. Before that I will once more write down how the axes of coordinates in system III are orientated.

We have taken

the $U^{(s)}$ -axis	towards the point	$\alpha = 270^{\circ}.0$	$\delta = -15^{\circ}.0$
» $V^{(s)}$ -axis	»	$\alpha = 335^{\circ}.7$	$\delta = +56^{\circ}.9$
» $W^{(s)}$ -axis	»	$\alpha = 188^{\circ}.5$	$\delta = +28^{\circ}.7$

The angles between these axes and the axes of the velocity ellipsoid are as follows

$$U^{(s)} U' = 5^{\circ}.0 \quad V^{(s)} V' = 6^{\circ}.4 \quad W^{(s)} W' = 6^{\circ}.1$$

and the $W^{(s)}$ axis lies $2^{\circ}.5$ from the pole of the Milky Way as given by KOBOLD

The characteristics are now, for the sake of completeness taking along also those of the second order:

$$\begin{aligned} \mathfrak{P}_1^2 N_{200}^{(s)} &= 2.4243 \pm 0.0435 & \mathfrak{P}_1^2 N_{110}^{(s)} &= -0.0080 \pm 0.0363 \\ \mathfrak{P}_1^2 N_{020}^{(s)} &= 1.0860 \pm 0.0435 & \mathfrak{P}_1^2 N_{101}^{(s)} &= -0.1309 \pm 0.0391 \\ \mathfrak{P}_1^2 N_{002}^{(s)} &= 0.7482 \pm 0.0417 & \mathfrak{P}_1^2 N_{011}^{(s)} &= -0.0625 \pm 0.0391. \end{aligned}$$

$$\begin{aligned} \mathfrak{P}_1^3 B_{300}^{(s)} &= -0.1054 \pm 0.0224 & \mathfrak{P}_1^4 B_{400}^{(s)} &= -0.2763 \pm 0.0130 \\ \mathfrak{P}_1^3 B_{210}^{(s)} &= -0.0208 \pm 0.0408 & \mathfrak{P}_1^4 B_{310}^{(s)} &= -0.1646 \pm 0.0217 \\ \mathfrak{P}_1^3 B_{120}^{(s)} &= -0.0986 \pm 0.0408 & \mathfrak{P}_1^4 B_{220}^{(s)} &= -0.1447 \pm 0.0245 \\ \mathfrak{P}_1^3 B_{030}^{(s)} &= +0.0148 \pm 0.0296 & \mathfrak{P}_1^4 B_{130}^{(s)} &= -0.0779 \pm 0.0219 \\ \mathfrak{P}_1^3 B_{021}^{(s)} &= -0.0166 \pm 0.0348 & \mathfrak{P}_1^4 B_{040}^{(s)} &= -0.0108 \pm 0.0131 \\ \mathfrak{P}_1^3 B_{012}^{(s)} &= -0.0766 \pm 0.0366 & \mathfrak{P}_1^4 B_{031}^{(s)} &= -0.0780 \pm 0.0178 \\ \mathfrak{P}_1^3 B_{003}^{(s)} &= +0.0283 \pm 0.0299 & \mathfrak{P}_1^4 B_{022}^{(s)} &= -0.0455 \pm 0.0201 \\ \mathfrak{P}_1^3 B_{102}^{(s)} &= -0.1765 \pm 0.0366 & \mathfrak{P}_1^4 B_{013}^{(s)} &= +0.1424 \pm 0.0201 \\ \mathfrak{P}_1^3 B_{201}^{(s)} &= -0.0448 \pm 0.0348 & \mathfrak{P}_1^4 B_{004}^{(s)} &= +0.0509 \pm 0.0159 \\ \mathfrak{P}_1^3 B_{111}^{(s)} &= +0.0229 \pm 0.0510 & \mathfrak{P}_1^4 B_{103}^{(s)} &= -0.0494 \pm 0.0201 \\ & & \mathfrak{P}_1^4 B_{202}^{(s)} &= -0.1465 \pm 0.0201 \\ & & \mathfrak{P}_1^4 B_{301}^{(s)} &= -0.0440 \pm 0.0178 \\ & & \mathfrak{P}_1^4 B_{211}^{(s)} &= +0.0316 \pm 0.0315 \\ & & \mathfrak{P}_1^4 B_{121}^{(s)} &= +0.1099 \pm 0.0315 \\ & & \mathfrak{P}_1^4 B_{112}^{(s)} &= +0.0860 \pm 0.0307 \end{aligned}$$

For the β characteristics we have the following values:

$$\begin{aligned} r_{12} &= -0.0049 \pm 0.0223 & \beta_{400}^{(s)} &= -0.0470 \pm 0.0022 \\ r_{13} &= -0.1021 \pm 0.0290 & \beta_{310}^{(s)} &= -0.0418 \pm 0.0055 \\ r_{23} &= -0.0693 \pm 0.0433 & \beta_{220}^{(s)} &= -0.0550 \pm 0.0093 \\ & & \beta_{130}^{(s)} &= -0.0441 \pm 0.0124 \\ \beta_{300}^{(s)} &= -0.0279 \pm 0.0059 & \beta_{040}^{(s)} &= -0.0092 \pm 0.0111 \\ \beta_{210}^{(s)} &= -0.0082 \pm 0.0161 & \beta_{031}^{(s)} &= -0.0797 \pm 0.0181 \\ \beta_{120}^{(s)} &= -0.0583 \pm 0.0241 & \beta_{022}^{(s)} &= -0.0560 \pm 0.0247 \\ \beta_{030}^{(s)} &= +0.0130 \pm 0.0261 & \beta_{013}^{(s)} &= +0.2111 \pm 0.0298 \\ \beta_{021}^{(s)} &= -0.0177 \pm 0.0370 & \beta_{004}^{(s)} &= +0.0909 \pm 0.0284 \\ \beta_{012}^{(s)} &= -0.0982 \pm 0.0469 & \beta_{103}^{(s)} &= -0.0490 \pm 0.0199 \\ \beta_{003}^{(s)} &= +0.0437 \pm 0.0462 & \beta_{202}^{(s)} &= -0.0808 \pm 0.0111 \\ \beta_{102}^{(s)} &= -0.1515 \pm 0.0314 & \beta_{301}^{(s)} &= -0.0135 \pm 0.0055 \\ \beta_{201}^{(s)} &= -0.0214 \pm 0.0166 & \beta_{211}^{(s)} &= +0.0145 \pm 0.0144 \\ \beta_{111}^{(s)} &= +0.0163 \pm 0.0363 & \beta_{121}^{(s)} &= +0.0751 \pm 0.0215 \\ & & \beta_{112}^{(s)} &= -0.0708 \pm 0.0252 \end{aligned}$$

These latter characteristics are as we have before shown practically independent of ϑ_1 and its systematical variation. *With their mean errors they give a general and complete description of the distribution of the linear motions of the stars of a magnitude brighter than 6.0 as far as it is given by the proper motions and, as far as it is possible at present to determine the value of the constant q' .*

30. We are now in a position to ascertain in how much our results are in consistence with the *two-stream hypothesis*

In the following table we give the values of the characteristics as computed from the figures cited from Eddington in § 25. In the second we point out what becomes of the characteristics if we assume that the internal spread of motion is different in the two streams. In the third column we give the evidence of our own values.

	Eddington	Generalized two-stream hyp.	Our values
$\sigma_1^{(s)} : \sigma_2^{(s)}$	1.63	—	1.50 ± 0.05
$\sigma_1^{(s)} : \sigma_8^{(s)}$	1.63	—	1.66 ± 0.05
$\sigma_2^{(s)} : \sigma_8^{(s)}$	1.00	—	1.11 ± 0.03
$\beta_{200}^{(s)}$	— 0.033	—	$- 0.027 \pm 0.006$
$\beta_{210}^{(s)}$	0.000	0.000	within mean error
$\beta_{120}^{(s)}$	0.000	pos. or neg.	slightly neg.
$\beta_{080}^{(s)}$	0.000	0.000	within mean error
$\beta_{021}^{(s)}$	0.000	0.000	, , ,
$\beta_{012}^{(s)}$	0.000	0.000	slightly negative
$\beta_{008}^{(s)}$	0.000	0.000	within mean error
$\beta_{102}^{(s)}$	0.000	pos. or neg.	neg.
$\beta_{201}^{(s)}$	0.000	0.000	slightly neg.
$\beta_{111}^{(s)}$	—	—	— —
$\beta_{490}^{(s)}$	— 0.030	—	$- 0.047 \pm 0.002$
$\beta_{810}^{(s)}$	0.000	0.000	neg.
$\beta_{220}^{(s)}$	0.000	pos.	neg.
$\beta_{130}^{(s)}$	0.000	0.000	neg.
$\beta_{040}^{(s)}$	0.000	pos.	within mean error
$\beta_{031}^{(s)}$	0.000	0.000	neg.
$\beta_{022}^{(s)}$	0.000	pos.	slightly neg.
$\beta_{013}^{(s)}$	0.000	0.000	strongly pos.
$\beta_{004}^{(s)}$	0.000	pos.	pos.
$\beta_{103}^{(s)}$	0.300	0.000	neg.
$\beta_{292}^{(s)}$	0.000	pos.	neg.
$\beta_{301}^{(s)}$	0.000	0.000	slightly neg.
$\beta_{211}^{(s)}$	—	—	— —
$\beta_{121}^{(s)}$	—	—	— —
$\beta_{112}^{(s)}$	—	—	— —

As a conclusion we may say that the two-stream hypothesis seems to be inadequate as a *physical explanation* of the properties of distribution of stellar movements. As a *working hypothesis* it is as good as the assumption of an ellipsoid of revolution, besides giving a pretty good description of the *skewness*. As regards the *excess* the two-stream hypothesis cannot be thought to give even its main features. If it should be necessary to adopt another value of q' the two-stream distribution will give a still poorer fit.

31. In figures 2, 3 and 4 we have plotted out the *frequency curves* of the motion as projected on the $U^{(s)}$ - the $V^{(s)}$ - and the $W^{(s)}$ -axes; the dotted curve is the normal frequency curve and the unity is the respective dispersions.

In a similar way fig. 5, 6 and 7 illustrate the distribution of the *apparent motions* as projected on the same axes. It is seen by the form of those curves that in the case of the *apparent* motions the characteristics of orders higher than the fourth cannot be neglected.

On the whole the fig. 2--7 and the other results of this chapter and chapter VII show the remarkable and very interesting fact, that in spite of the distribution of the apparent motions being so excessively unnormal as to make its representation by a frequency function of the A-type almost illusory, or at least requiring the computations of characteristics of the fifth and sixth orders, the distribution of the linear motions may well be thought to be a nearly normal three axial distribution.

On speaking with MR. GYLLENBERG of this Observatory on the above subjects he has told me that the information to be gathered from the radial velocities in all essentials is in consistence with the conclusions here drawn. Taking together stars surrounding each vertex he has computed frequency curves that may be regarded as giving the essentials of the distribution along our axes of coordinates. The only deviation from our results were found to lie in the excess' along the $U^{(s)}$ and $V^{(s)}$ axes. In the $U^{(s)}$ axis he found the excess to be zero and in the $V^{(s)}$ axis to be about 10 % positive ($\beta_{040} = +0.04$); the main feature of the higher characteristics of the radial velocities is, however, that they are all small. Knowing what would become of our characteristics if another value of q' should be adopted, we may assert, that *the value $q' = 0.75$ gives the best agreement possible to obtain between the higher characteristics as derived from the proper motions and those indicated by the radial velocities.*

As is well known Prof. CHARLIER has in his here often cited work concluded that the skewness of the linear motions is not small and that the excess is »thoroughly negative». Evidently this opinion is due to his having used a value of q' so low as 0.62. As seen from our tables XIV and XV of chapter VII $q' = 0.62$ gives most thoroughly negative values of the higher characteristics. To the same cause we also may attribute the circumstance that his attempts at dissecting the correlation function of the linear motions into two components, viz. the two star-streams, gave so altogether negative results. Evidently an attempt to reconcile the charac-

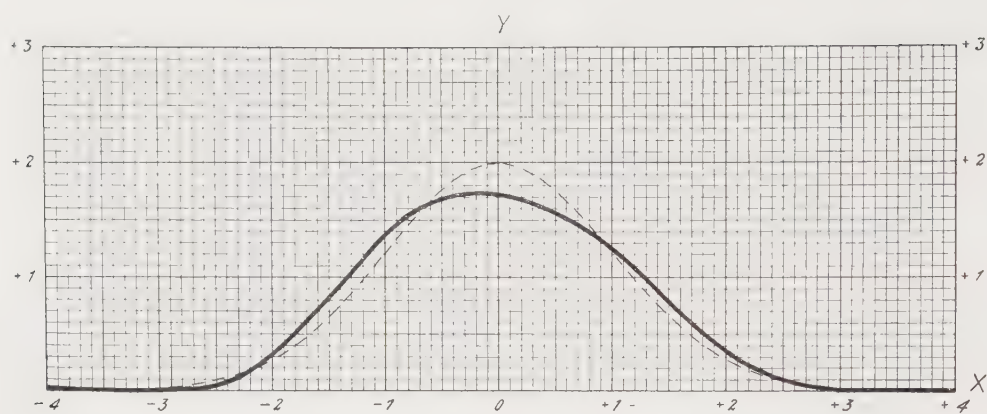


Fig. 2. Frequency curve of the *linear motions* as projected on the axis toward vertex I.

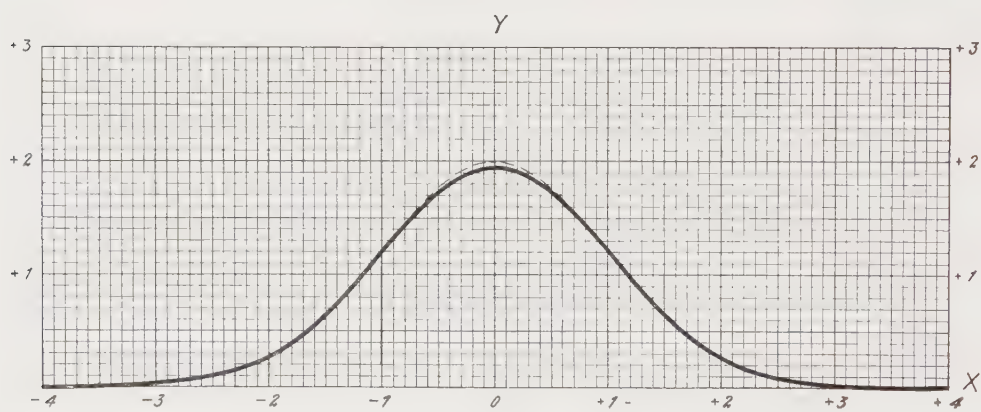


Fig. 3. Frequency curve of the *linear motions* as projected on the axis toward vertex II.

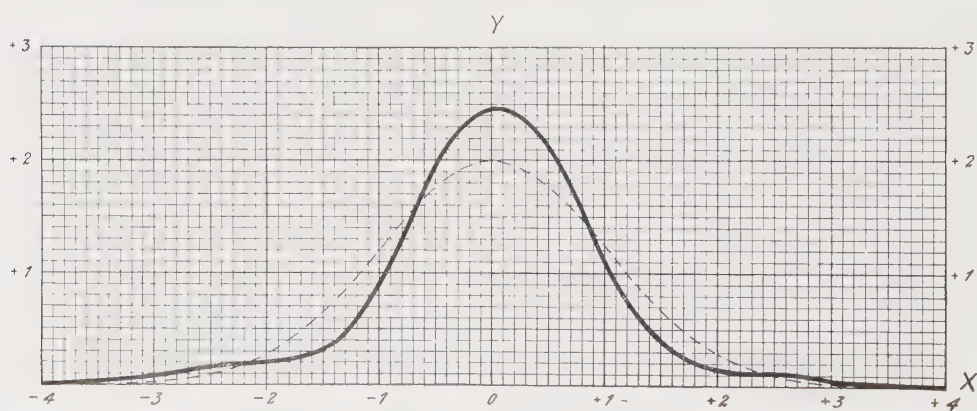


Fig. 4. Frequency curve of the *linear motions* as projected on the axis toward vertex III.

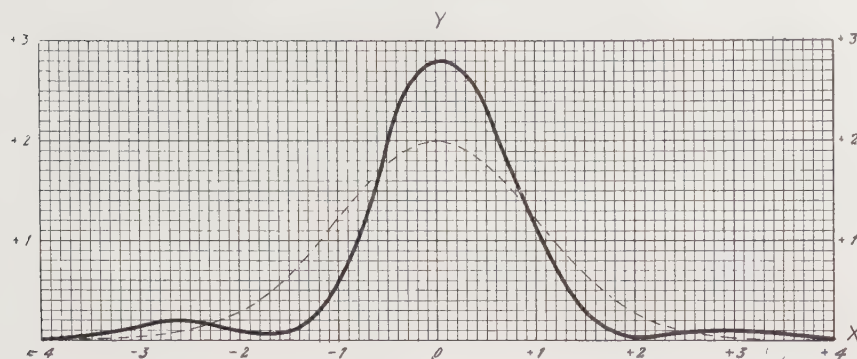


Fig. 5. Frequency curve of the *apparent motions* as projected on the axis toward vertex I.

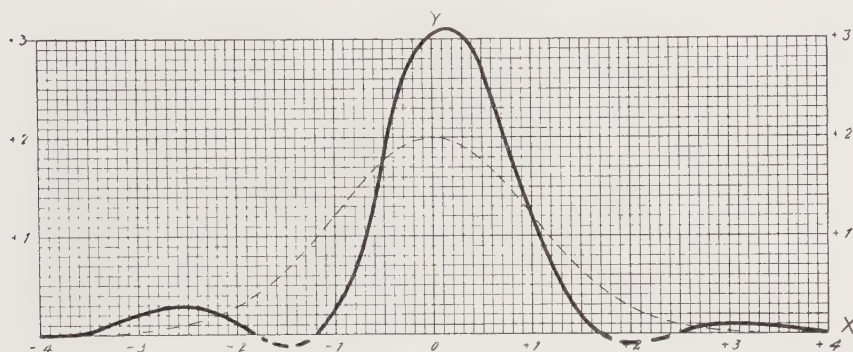


Fig. 6. Frequency curve of the *apparent motions* as projected on the axis toward vertex II.

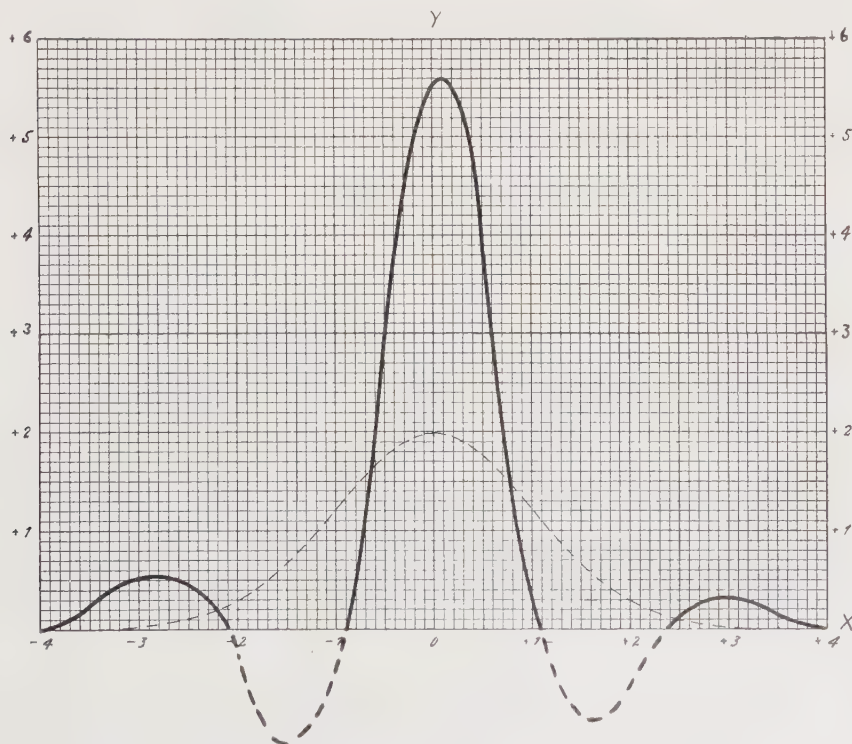


Fig. 7. Frequency curve of the *apparent motions* as projected on the axis toward vertex III.

teristics given in tables XIV and XV for $q' = 0.62$ with the two-stream hypothesis would be hopeless.

32. In the paragraphs 35 and 36 of his work CHARLIER makes some exceedingly interesting remarks regarding the bearing of the law of MAXWELL for the molecules of a gas on the distribution of stellar velocities. The law of MAXWELL requires that taking out particles (stars) of the same mass the distribution of their velocities in any one direction shall be of the normal form. What would then be the distribution when particles (stars) of different masses are mixed? Assuming the logarithmus of the mass to be normally distributed with the dispersion L , CHARLIER finds that the skewness should be zero and the excess given by

$$E = \frac{3}{8}(e^{L^2} - 1),$$

or taking the luminosity to be proportional to a constant power c of the mass

$$(87) \quad E = \frac{3}{8}(e^{\kappa^2 \sigma_s^2} - 1),$$

where

$$\kappa = \frac{0.921034}{c},$$

and σ_s is the dispersion of the absolute magnitudes.

To get an opinion of the order of magnitude of this excess we will proceed as follows:

Assuming for a moment the density of the stars to be of the form used by SEELIGER

$$D = \gamma r^{-t}$$

we should find from the equations of paragraph 11

$$\Delta_0(y) = Ke^{(t-3)y^2/2},$$

and by equations (47) and (47*) *

$$(50^*) \quad \vartheta_s(m) = \vartheta_1^s(m) \left(\frac{1}{q'} \right)^{1/2(s^2-s)},$$

and here q' has the form

$$q' = e^{-b^2 \sigma_s^2}.$$

By computations similar to those performed in § 12 we should find that for all stars brighter than 6.0 we have to take

$$q' = 0.88 e^{-b^2 \sigma_s^2}.$$

* It has already been pointed out by CHARLIER that the assumption of SEELIGER regarding the density function will give rise to the expression (50*) for ϑ_s .

Introducing this expression into equation (87) we find

$$E = \sqrt[3]{\left[\left(\frac{0.88}{q'}\right)^{\frac{\kappa^2}{b^2}} - 1\right]}.$$

Taking with CHARLIER $c = 3$ we find $\kappa = 0.3070$, and for $q' = 0.75$

$$E = + 0.026,$$

which corresponds to

$$\beta_4 = + 0.009.$$

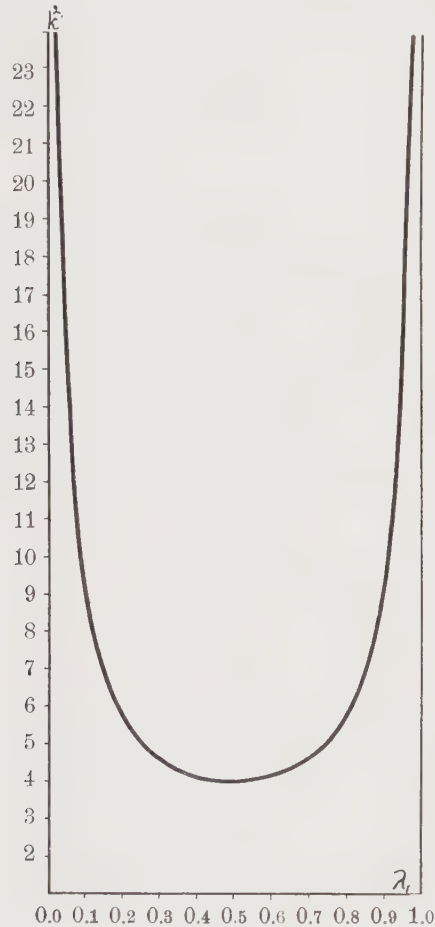


Fig. 8.

The parameter λ_1 as depending on the dispersion of the apparent magnitudes.

The smallest value acceptable for c is $2/3$. This value would make E about four times as great as the above value. As seen the Maxwellian law would give a very small and positive excess. I cannot say that our results are exactly in contradiction with the requirements of the law of Maxwell. Of course they, on the other side, give no positive evidence whatsoever.

33. We have in this investigation found that many circumstances point towards a value 0.75 for q' . A question of interest in stellar astronomy is: what value of the constant λ_1 comes out of this value for q' ?

We remind of the equation

$$q' = \alpha_1 e^{-(b^2 k^2 - \alpha_2) \lambda_1 (1 - \lambda_1)},$$

$$\alpha_1 = 0.88; \alpha_2 = 0.43.$$

Clearly the value of λ_1 depends on what value is to be taken for the dispersion of the apparent magnitudes k , of which, on the whole, we only know that it has a value of about 3 or 4.

For $q' = 0.75$ the above equation may be written

$$k^2 - 2.03 = \frac{0.754}{\lambda_1 (1 - \lambda_1)}.$$

Taking k^2 and λ_1 as coordinates of a Cartesian system this equation is represented by the curve drawn up in fig. 8, from which it is seen that for $k = 3.0$ we have $\lambda_1 = 0.87$ and for $k = 4$, $\lambda_1 = 0.94$. Now the values of the factor 0.88 and the term 0.43 depends on our having regarded the stars brighter than 6.0 as being chiefly of magnitudes fainter than 3.0. As it has been necessary to have a lower limit of the magnitudes when discussing the relation between λ_1 and q' , it is impossible to say if it is chosen too high or too low. Taking a higher limit, α_1 will be found greater and α_2 smaller and *vice versa*. Thus if the lower limit is taken higher up, the curve will be lifted upwards, and taking a yet lower limit the curve will lie nearer to the λ_1 -axis. The effect on the order of magnitude of λ_1 for k about 3 and 4 will however be small and hence we conclude that λ_1 has a value of about 0.9.

Before finishing this memoir I grasp the opportunity to express my sincere gratitude to Professor C. V. L. CHARLIER for the many fertile impulses that I have received during the planning and preparation of this investigation by his always intent interest for the questions here inquired into. Especially I wish also to express my heartiest thanks for the kind readiness with which he has placed at my disposal as yet unpublished material of a character to greatly facilitate my work.

It is further with sincere pleasure that I acknowledge the obligations in which I stand to Mr. K. A. W. GYLLENBERG for many useful suggestions received during our daily conversations on these and related subjects and especially for his kindness to permit me to use the results of his own investigations of the radial velocities.

To the Royal Physiographic Society of Lund I also wish to express my thanks for pecuniary support granted me out of the Anders Jahan Retzius' Minnesfond.

TABLE II.
Equations of transformation for the ellipsoid.

	α_1	α_2	α_3	α_4	α_5	α_6	ν_{02}	γ_0^2
A_1	0.0000	+ 0.9698	+ 0.0301	0.0000	- 0.3420	0.0000	+ 2.594	+ 0.391
A_2	0.0000	+ 0.9698	+ 0.0301	0.0000	+ 0.3420	0.0000	+ 3.038	+ 0.759
B_1	+ 0.4539	+ 0.0479	+ 0.4983	+ 0.2950	- 0.3090	- 0.9512	+ 1.124	+ 0.328
B_2	+ 0.1733	+ 0.3233	+ 0.4983	+ 0.4772	- 0.8090	- 0.5878	+ 1.103	+ 0.555
B_3	0.0000	+ 0.5017	+ 0.4983	0.0000	- 1.0000	0.0000	+ 2.042	+ 0.939
B_4	+ 0.1733	+ 0.3233	+ 0.4983	- 0.4772	- 0.8090	+ 0.5878	+ 1.458	+ 0.450
B_5	+ 0.4539	+ 0.0479	+ 0.4983	- 0.2950	- 0.3090	+ 0.9512	+ 2.411	+ 0.407
B_6	+ 0.4539	+ 0.0479	+ 0.4983	+ 0.2950	+ 0.3090	+ 0.9512	+ 1.265	+ 0.060
B_7	+ 0.1733	+ 0.3233	+ 0.4983	+ 0.4772	+ 0.8090	+ 0.5878	+ 3.680	+ 0.051
B_8	0.0000	+ 0.5017	+ 0.4983	0.0000	+ 1.0000	0.0000	+ 3.172	+ 0.089
B_9	+ 0.1733	+ 0.3233	+ 0.4983	- 0.4772	+ 0.8090	- 0.5878	+ 1.508	+ 0.000
B_{10}	+ 0.4539	+ 0.0479	+ 0.4983	- 0.2950	+ 0.3090	- 0.9512	+ 0.918	+ 0.023
C_1	+ 0.0583	+ 0.0042	+ 0.9376	+ 0.0312	- 0.1250	- 0.4676	+ 1.553	+ 0.440
C_2	+ 0.0313	+ 0.0313	+ 0.9376	+ 0.0625	- 0.3424	- 0.3424	+ 1.458	+ 0.601
C_3	+ 0.0042	+ 0.0583	+ 0.9376	+ 0.0312	- 0.4676	- 0.1250	+ 0.884	+ 0.501
C_4	+ 0.0042	+ 0.0583	+ 0.9376	- 0.0312	- 0.4676	+ 0.1250	+ 1.598	+ 0.487
C_5	+ 0.0313	+ 0.0313	+ 0.9376	- 0.0625	- 0.3424	+ 0.3424	+ 2.489	+ 0.924
C_6	+ 0.0583	+ 0.0042	+ 0.9376	- 0.0312	- 0.1250	+ 0.4676	+ 1.218	+ 0.399
C_7	+ 0.0583	+ 0.0042	+ 0.9376	+ 0.0312	+ 0.1250	+ 0.4676	+ 2.334	+ 0.162
C_8	+ 0.0313	+ 0.0313	+ 0.9376	+ 0.0625	+ 0.3424	+ 0.3424	+ 1.584	+ 0.332
C_9	+ 0.0042	+ 0.0583	+ 0.9376	+ 0.0312	+ 0.4676	+ 0.1250	+ 1.617	+ 0.104
C_{10}	+ 0.0042	+ 0.0583	+ 0.9376	- 0.0312	+ 0.4676	- 0.1250	+ 1.513	+ 0.039
C_{11}	+ 0.0313	+ 0.0313	+ 0.9376	- 0.0625	+ 0.3424	- 0.3424	+ 1.601	+ 0.225
C_{12}	+ 0.0583	+ 0.0042	+ 0.9376	- 0.0312	+ 0.1250	- 0.4676	+ 2.018	+ 0.027

	β_1	β_2	β_3	β_4	β_5	β_6	ν_{20}	x_0^2
A_1	+ 1.0000	0.0000		0.0000			+ 1.245	+ 0.032
A_2	+ 1.0000	0.0000		0.0000			+ 0.739	+ 0.002
B_1	+ 0.0954	+ 0.9046		- 0.5878			+ 2.542	+ 0.478
B_2	+ 0.6545	+ 0.3455		- 0.9510			+ 1.111	+ 0.259
B_3	+ 1.0000	0.0000		0.0000			+ 0.651	+ 0.023
B_4	+ 0.6545	+ 0.3455		+ 0.9510			+ 1.456	+ 0.242
B_5	+ 0.0954	+ 0.9046		+ 0.5878			+ 3.110	+ 0.984
B_6	+ 0.0954	+ 0.9046		- 0.5878			+ 2.674	+ 0.500
B_7	+ 0.6545	+ 0.3455		- 0.9510			+ 3.452	+ 0.554
B_8	+ 1.0000	0.0000		0.0000			+ 1.353	+ 0.032
B_9	+ 0.6545	+ 0.3455		+ 0.9510			+ 0.922	+ 0.246
B_{10}	+ 0.0954	+ 0.9046		+ 0.5878			+ 1.855	+ 0.433
C_1	+ 0.0670	+ 0.9330		- 0.5000			+ 3.696	+ 0.554
C_2	+ 0.5000	+ 0.5000		- 1.0000			+ 2.900	+ 0.610
C_3	+ 0.9330	+ 0.0670		- 0.5000			+ 1.445	+ 0.242
C_4	+ 0.9330	+ 0.0670		+ 0.5000			+ 1.240	+ 0.062
C_5	+ 0.5000	+ 0.5000		+ 1.0000			+ 2.093	+ 0.566
C_6	+ 0.0670	+ 0.9330		+ 0.5000			+ 4.681	+ 0.702
C_7	+ 0.0670	+ 0.9330		- 0.5000			+ 2.890	+ 0.623
C_8	+ 0.5000	+ 0.5000		- 1.0000			+ 3.946	+ 0.696
C_9	+ 0.9330	+ 0.0670		- 0.5000			+ 1.361	+ 0.062
C_{10}	+ 0.9330	+ 0.0670		+ 0.5000			+ 1.065	+ 0.102
C_{11}	+ 0.5000	+ 0.5000		+ 1.0000			+ 2.522	+ 0.387
C_{12}	+ 0.0670	+ 0.9330		+ 0.5000			+ 3.430	+ 1.067

	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_{11}	$x_0 y_0$
A_1	0.0000	0.0000		+ 0.9843	0.0000	— 0.1738	— 0.589	— 0.112
A_2	0.0000	0.0000		+ 0.9848	0.0000	+ 0.1736	+ 0.270	+ 0.040
B_1	+ 0.2082	— 0.2082		— 0.5731	+ 0.6713	— 0.2181	— 0.164	— 0.399
B_2	+ 0.3368	— 0.3368		+ 0.2189	+ 0.4149	— 0.5711	— 0.591	— 0.379
B_3	0.0000	0.0000		+ 0.7083	0.0000	— 0.7059	+ 0.065	— 0.556
B_4	— 0.3368	+ 0.3368		+ 0.2189	— 0.4149	— 0.5711	+ 0.594	+ 0.330
B_5	— 0.2082	+ 0.2082		— 0.5731	— 0.6713	— 0.2181	+ 0.233	+ 0.633
B_6	+ 0.2082	— 0.2082		— 0.5731	— 0.6713	+ 0.2181	— 0.355	+ 0.173
B_7	+ 0.3368	— 0.3368		+ 0.2189	— 0.4149	+ 0.5711	— 1.609	— 0.167
B_8	0.0000	0.0000		+ 0.7083	0.0000	+ 0.7059	+ 0.457	+ 0.054
B_9	— 0.3368	+ 0.3368		+ 0.2189	+ 0.4149	+ 0.5711	+ 0.627	— 0.009
B_{10}	— 0.2082	+ 0.2082		— 0.5731	+ 0.6713	+ 0.2181	+ 0.614	— 0.101
C_1	+ 0.0625	— 0.0625		— 0.2165	+ 0.9353	— 0.2506	+ 0.167	— 0.493
C_2	+ 0.1250	— 0.1250		0.0000	+ 0.6847	— 0.6847	+ 0.357	— 0.605
C_3	+ 0.0625	— 0.0625		+ 0.2165	+ 0.2506	— 0.9353	+ 0.136	— 0.348
C_4	— 0.0625	+ 0.0625		+ 0.2165	— 0.2506	— 0.9353	— 0.158	+ 0.174
C_5	— 0.1250	+ 0.1200		0.0000	— 0.6847	— 0.6847	— 0.057	+ 0.722
C_6	— 0.0625	+ 0.0625		— 0.2165	— 0.9353	— 0.2506	— 0.450	+ 0.530
C_7	+ 0.0625	— 0.0625		— 0.2165	— 0.9353	+ 0.2506	— 0.753	+ 0.318
C_8	+ 0.1250	— 0.1250		0.0000	— 0.6847	+ 0.6847	+ 0.091	+ 0.480
C_9	+ 0.0625	— 0.0625		+ 0.2165	— 0.2506	+ 0.9353	— 0.458	+ 0.080
C_{10}	— 0.0625	+ 0.0625		+ 0.2165	+ 0.2506	+ 0.9353	— 0.151	— 0.063
C_{11}	— 0.1250	+ 0.1250		0.0000	+ 0.6847	+ 0.6847	+ 0.093	— 0.295
C_{12}	— 0.0625	+ 0.0625		— 0.2165	+ 0.9353	+ 0.2506	+ 0.135	— 0.169

TABLE III.

Equations of transformation for the coefficients of the third order.

	$-a_1$	$-a_2$	a_3	$-a_4$	a_5	$-a_6$	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}
A_1	+ 1.0000	0.0000		0.0000		0.0000					- 0.2415	+ 0.112	+ 0.0010
A_2	- 1.0000	0.0000		0.0000		0.0000					+ 0.1003	+ 0.017	0.0000
B_1	+ 0.0295	- 0.8604		- 0.0908		+ 0.2795					- 0.9455	+ 0.886	+ 0.0554
B_2	+ 0.5295	- 0.2030		- 0.3847		+ 0.2795					- 0.3297	+ 0.283	+ 0.0228
B_3	+ 1.0000	0.0000		0.0000		0.0000					- 0.0002	+ 0.053	+ 0.0007
B_4	+ 0.5295	+ 0.2030		+ 0.3847		+ 0.2795					+ 0.0532	- 0.358	+ 0.0198
B_5	+ 0.0295	+ 0.8604		+ 0.0908		+ 0.2795					+ 0.7733	- 1.543	+ 0.1627
B_6	- 0.0295	+ 0.8604		+ 0.0908		- 0.2795					+ 0.1142	- 0.946	+ 0.0589
B_7	- 0.5295	+ 0.2030		+ 0.3847		- 0.2795					+ 1.5635	- 1.234	+ 0.0687
B_8	- 1.0000	0.0000		0.0000		0.0000					- 0.5115	+ 0.122	- 0.0010
B_9	- 0.5295	- 0.2030		- 0.3847		- 0.2795					- 0.3275	+ 0.229	- 0.0203
B_{10}	- 0.0295	- 0.8604		- 0.0908		- 0.2795					- 0.1910	+ 0.611	- 0.0475
C_1	+ 0.0173	- 0.9011		- 0.0647		+ 0.2415					+ 0.8357	+ 1.375	- 0.0687
C_2	+ 0.3535	- 0.3535		- 0.3535		+ 0.3535					- 0.3948	+ 1.133	- 0.0794
C_3	+ 0.9011	- 0.0173		- 0.2415		+ 0.0647					+ 0.0448	+ 0.356	- 0.0198
C_4	+ 0.9011	+ 0.0173		+ 0.2415		+ 0.0647					- 0.2165	- 0.155	+ 0.0020
C_5	+ 0.3535	+ 0.3535		+ 0.3535		+ 0.3535					+ 0.6308	- 0.787	+ 0.0709
C_6	+ 0.0173	+ 0.9011		+ 0.0647		+ 0.2415					+ 0.3972	- 1.962	+ 0.0981
C_7	- 0.0173	+ 0.9011		+ 0.0647		- 0.2415					+ 0.2242	- 1.140	+ 0.0819
C_8	- 0.3535	+ 0.3535		+ 0.3535		- 0.3535					+ 0.8599	- 1.646	+ 0.0967
C_9	- 0.9011	+ 0.0173		+ 0.2415		- 0.0647					- 0.0513	- 0.169	+ 0.0020
C_{10}	- 0.9011	- 0.0173		- 0.2415		- 0.0647					- 0.2710	+ 0.172	- 0.0106
C_{11}	- 0.3535	- 0.3535		- 0.3535		- 0.3535					- 0.4205	+ 0.785	- 0.0401
C_{12}	- 0.0173	- 0.9011		- 0.0647		- 0.2415					- 0.9130	+ 1.772	- 0.1837

	$-b_1$	$-b_2$	b_3	$-b_4$	$-b_5$	$-b_6$	$-b_7$	b_8	b_9	$-b_{10}$	b_{11}	b_{12}	b_{13}
A_1	0.0000	0.0000		+ 0.9848	- 0.1736	0.0000	0.0000			0.0000	+ 0.5175	- 0.495	+ 0.0100
A_2	0.0000	0.0000		- 0.9848	- 0.1736	0.0000	0.0000			0.0000	+ 0.1030	+ 0.335	- 0.0009
B_1	+ 0.1929	+ 0.5940		- 0.3751	- 0.0674	+ 0.4808	- 0.6386			+ 0.2075	+ 0.5650	- 0.843	+ 0.1369
B_2	+ 0.8175	+ 0.5940		- 0.0210	- 0.4620	- 0.4012	- 0.2439			+ 0.3357	+ 0.6545	- 0.715	+ 0.0869
B_3	0.0000	+ 0.0000		+ 0.7083	- 0.7059	0.0000	0.0000			0.0000	+ 0.2540	- 0.305	+ 0.0126
B_4	- 0.8175	+ 0.5940		- 0.0210	- 0.4620	+ 0.4012	- 0.2439			- 0.3357	- 0.0675	- 0.782	+ 0.0812
B_5	- 0.1929	+ 0.5940		- 0.3751	- 0.0674	- 0.4808	- 0.6386			- 0.2075	+ 0.1635	- 0.723	+ 0.3159
B_6	- 0.1929	- 0.5940		+ 0.3751	- 0.0674	- 0.4808	- 0.6386			+ 0.2075	- 0.2585	+ 0.136	+ 0.0612
B_7	- 0.8175	- 0.0940		+ 0.0210	- 0.4620	+ 0.4012	- 0.2439			+ 0.3357	- 2.1745	+ 1.586	- 0.0626
B_8	0.0000	0.0000		- 0.7083	- 0.7059	0.0000	0.0000			0.0000	- 0.2450	+ 0.284	- 0.0048
B_9	+ 0.8175	- 0.5940		+ 0.0210	- 0.4620	- 0.4012	- 0.2439			- 0.3357	- 0.8935	+ 0.302	+ 0.0023
B_{10}	+ 0.1929	- 0.5940		+ 0.3751	- 0.0674	+ 0.4808	- 0.6386			- 0.2075	+ 0.4515	+ 0.262	+ 0.0833
C_1	+ 0.0486	+ 0.1812		- 0.1165	- 0.0649	+ 0.1929	- 0.9034			+ 0.2421	+ 1.6205	- 1.614	+ 0.1839
C_2	+ 0.2652	+ 0.2652		- 0.0884	- 0.4842	- 0.0884	- 0.4842			+ 0.4842	+ 0.2010	- 0.856	+ 0.2364
C_3	+ 0.1812	+ 0.0486		+ 0.1929	- 0.9034	- 0.1165	- 0.0649			+ 0.2421	+ 0.8480	- 0.463	+ 0.0857
C_4	- 0.1812	+ 0.0486		+ 0.1929	- 0.9034	+ 0.1165	- 0.0649			- 0.2421	+ 0.3590	- 0.394	+ 0.0216
C_5	- 0.2652	+ 0.2652		- 0.0884	- 0.4842	+ 0.0884	- 0.4842			- 0.4842	+ 0.4335	- 0.963	+ 0.2336
C_6	- 0.0486	+ 0.1812		- 0.1165	- 0.0649	- 0.1929	- 0.9034			- 0.2421	+ 2.1330	- 1.102	+ 0.2218
C_7	- 0.0486	- 0.1812		+ 0.1165	- 0.0649	- 0.1929	- 0.9034			+ 0.2421	+ 0.3315	+ 0.018	+ 0.1258
C_8	- 0.2652	- 0.2652		+ 0.0884	- 0.4842	+ 0.0884	- 0.4842			+ 0.4842	+ 0.8020	- 1.207	+ 0.2004
C_9	- 0.1812	- 0.0486		- 0.1929	- 0.9034	+ 0.1165	- 0.0649			+ 0.2421	+ 0.7370	- 0.104	+ 0.0100
C_{10}	+ 0.1812	- 0.0486		- 0.1929	- 0.9034	- 0.1165	- 0.0649			- 0.2421	+ 0.5400	- 0.154	+ 0.0101
C_{11}	+ 0.2652	- 0.2652		+ 0.0884	- 0.4842	- 0.0884	- 0.4842			- 0.4842	+ 0.4020	- 0.533	+ 0.0920
C_{12}	+ 0.0486	- 0.1812		+ 0.1165	- 0.0649	+ 0.1929	- 0.9034			- 0.2421	- 0.4445	- 0.149	+ 0.0875

	$-c_1$	$-c_2$	c_3	$-c_4$	$-c_5$	$-c_6$	$-c_7$
A_1	0.0000	0.0000		0.0000	0.0000	+ 0.9698	0.0000
A_2	0.0000	0.0000		0.0000	0.0000	- 0.9698	0.0000
B_1	+ 0.4206	- 0.1368		- 0.3405	- 0.2938	- 0.2656	+ 0.2939
B_2	+ 0.4206	- 0.5790		+ 0.2841	- 0.4754	- 0.0148	+ 0.4754
B_3	0.0000	0.0000		0.0000	0.0000	+ 0.5017	0.0000
B_4	+ 0.4206	+ 0.5790		- 0.2841	+ 0.4754	- 0.0148	- 0.4754
B_5	+ 0.4206	+ 0.1368		+ 0.3405	+ 0.2938	- 0.2656	- 0.2939
B_6	- 0.4206	+ 0.1368		+ 0.3405	- 0.2938	+ 0.2656	+ 0.2939
B_7	- 0.4206	+ 0.5790		- 0.2841	- 0.4754	+ 0.0148	+ 0.4754
B_8	0.0000	0.0000		0.0000	0.0000	- 0.5017	0.0000
B_9	- 0.4206	- 0.5790		+ 0.2841	+ 0.4754	+ 0.0148	- 0.4754
B_{10}	- 0.4206	- 0.1368		- 0.3405	+ 0.2938	+ 0.2656	- 0.2939
C_1	+ 0.0453	- 0.0122		- 0.0481	- 0.1210	- 0.0291	+ 0.1210
C_2	+ 0.0663	- 0.0663		+ 0.0221	- 0.2421	- 0.0221	+ 0.2421
C_3	+ 0.0122	- 0.0453		+ 0.0291	- 0.1210	+ 0.0481	+ 0.1210
C_4	+ 0.0122	+ 0.0453		- 0.0291	+ 0.1210	+ 0.0481	- 0.1210
C_5	+ 0.0663	+ 0.0663		- 0.0221	+ 0.2421	+ 0.0221	- 0.2421
C_6	+ 0.0453	+ 0.0122		+ 0.0481	+ 0.1210	- 0.0291	- 0.1210
C_7	- 0.0453	+ 0.0122		+ 0.0481	- 0.1210	+ 0.0291	+ 0.1210
C_8	- 0.0663	+ 0.0663		- 0.0221	- 0.2421	+ 0.0221	+ 0.2421
C_9	- 0.0123	+ 0.0453		- 0.0291	- 0.1210	- 0.0481	+ 0.1210
C_{10}	- 0.0123	- 0.0453		+ 0.0291	+ 0.1210	- 0.0481	- 0.1210
C_{11}	- 0.0663	- 0.0663		+ 0.0221	+ 0.2421	+ 0.0221	- 0.2421
C_{12}	- 0.0453	- 0.0122		- 0.0481	+ 0.1210	+ 0.0291	- 0.1210

	$-c_8$	$-c_9$	$-c_{10}$	c_{11}	c_{12}	c_{13}
A_1	+ 0.0301	0.0000	- 0.1710	- 0.7915	+ 0.602	- 0.0352
A_2	- 0.0301	0.0000	- 0.1710	+ 0.2545	+ 0.307	- 0.0162
B_1	+ 0.1540	- 0.4739	+ 0.4046	+ 0.8605	+ 0.486	- 0.1143
B_2	+ 0.4031	- 0.2929	- 0.1545	- 0.9745	+ 0.721	- 0.1415
B_3	+ 0.4983	0.0000	- 0.5000	+ 0.2505	+ 0.102	- 0.0760
B_4	+ 0.4031	+ 0.2929	- 0.1545	+ 0.1280	- 0.757	+ 0.1107
B_5	+ 0.1540	+ 0.4739	+ 0.4046	+ 0.7040	- 0.344	+ 0.2020
B_6	- 0.1540	+ 0.4739	+ 0.4046	+ 0.3785	- 0.287	+ 0.0212
B_7	- 0.4031	+ 0.2929	- 0.1545	+ 2.1685	- 1.731	+ 0.0190
B_8	- 0.4983	0.0000	- 0.5000	- 0.3745	+ 0.422	- 0.0080
B_9	- 0.4031	- 0.2929	- 0.1545	- 1.1690	+ 0.362	0.0000
B_{10}	- 0.1540	- 0.4739	+ 0.4046	- 0.1485	+ 0.208	- 0.0075
C_1	+ 0.2427	- 0.9057	+ 0.2096	+ 0.3540	+ 0.489	- 0.1716
C_2	+ 0.6630	- 0.6630	0.0000	- 0.8890	+ 0.308	- 0.2346
C_3	+ 0.9057	- 0.2427	- 0.2096	+ 0.1915	+ 0.134	- 0.1232
C_4	+ 0.9057	+ 0.2427	- 0.2096	- 0.8275	- 0.089	+ 0.0609
C_5	+ 0.6630	+ 0.6630	0.0000	+ 0.1025	- 0.881	+ 0.3474
C_6	+ 0.2427	+ 0.9057	+ 0.2096	- 0.3365	- 0.226	+ 0.1676
C_7	- 0.2427	+ 0.9057	+ 0.2096	- 0.1895	- 0.614	+ 0.2315
C_8	- 0.6630	+ 0.6630	0.0000	+ 0.3250	- 0.709	+ 0.1384
C_9	- 0.9057	+ 0.2427	- 0.2096	- 0.1440	- 0.052	+ 0.0129
C_{10}	- 0.9057	- 0.2427	- 0.2096	- 0.5385	+ 0.272	- 0.0062
C_{11}	- 0.6630	- 0.6630	0.0000	+ 0.0845	+ 0.446	- 0.0700
C_{12}	- 0.2427	- 0.9057	+ 0.2096	- 0.0425	+ 0.022	- 0.0139

	$-d_1$	$-d_2$	$-d_3$	$-d_4$	$-d_5$	$-d_6$	$-d_7$
A_1	0.0000	+ 0.9551	- 0.0052	0.0000	0.0000	- 0.0000	- 0.1683
A_2	0.0000	- 0.9551	- 0.0052	0.0000	0.0000	0.0000	- 0.1683
B_1	+ 0.3058	+ 0.0105	- 0.3517	+ 0.0994	- 0.3204	+ 0.0323	- 0.0333
B_2	+ 0.0721	+ 0.1881	- 0.3517	+ 0.0993	- 0.1223	+ 0.1367	- 0.2317
B_3	0.0000	+ 0.3533	- 0.3517	0.0000	0.0000	0.0000	- 0.3542
B_4	- 0.0721	+ 0.1881	- 0.3517	+ 0.0993	- 0.1223	- 0.1367	- 0.2317
B_5	- 0.3058	+ 0.0105	- 0.3517	+ 0.0993	- 0.3204	- 0.0323	- 0.0338
B_6	- 0.3058	- 0.0105	- 0.3517	- 0.0993	- 0.3204	- 0.0323	- 0.0338
B_7	- 0.0721	- 0.1881	- 0.3517	- 0.0993	- 0.1223	- 0.1367	- 0.2317
B_8	0.0000	- 0.3533	- 0.3517	0.0000	0.0000	0.0000	- 0.3542
B_9	+ 0.0721	- 0.1881	- 0.3517	- 0.0993	- 0.1223	+ 0.1367	- 0.2317
B_{10}	+ 0.3058	- 0.0105	- 0.3517	- 0.0993	- 0.3204	+ 0.0323	- 0.0338
C_1	+ 0.0141	+ 0.0003	- 0.9079	+ 0.0038	- 0.0565	+ 0.0010	- 0.0041
C_2	+ 0.0055	+ 0.0055	- 0.9079	+ 0.0055	- 0.0303	+ 0.0055	- 0.0303
C_3	+ 0.0003	+ 0.0141	- 0.9079	+ 0.0010	- 0.0041	+ 0.0033	- 0.0565
C_4	- 0.0003	+ 0.0141	- 0.9079	+ 0.0010	- 0.0041	- 0.0033	- 0.0565
C_5	- 0.0055	+ 0.0055	- 0.9079	+ 0.0055	- 0.0303	- 0.0055	- 0.0303
C_6	- 0.0141	+ 0.0003	- 0.9079	+ 0.0038	- 0.0565	- 0.0010	- 0.0041
C_7	- 0.0141	- 0.0003	- 0.9079	- 0.0038	- 0.0565	- 0.0010	- 0.0041
C_8	- 0.0055	- 0.0055	- 0.9079	- 0.0055	- 0.0303	- 0.0055	- 0.0303
C_9	- 0.0003	- 0.0141	- 0.9079	- 0.0010	- 0.0041	- 0.0033	- 0.0565
C_{10}	+ 0.0003	- 0.0141	- 0.9079	- 0.0010	- 0.0041	+ 0.0033	- 0.0565
C_{11}	+ 0.0055	- 0.0055	- 0.9079	- 0.0055	- 0.0303	+ 0.0055	- 0.0303
C_{12}	+ 0.0141	- 0.0003	- 0.9079	- 0.0038	- 0.0565	+ 0.0010	- 0.0041

	$-d_8$	$-d_9$	$-d_{10}$	d_{11}	d_{12}	d_{13}
A_1	0.0000	+ 0.0296	0.0000	+ 0.4345	- 0.811	+ 0.0407
A_2	0.0000	- 0.0296	0.0000	- 0.4345	+ 0.811	- 0.0407
B_1	+ 0.3357	+ 0.1091	- 0.1041	+ 0.0865	- 0.322	+ 0.0313
B_2	+ 0.2074	+ 0.2855	- 0.1684	+ 0.4820	- 0.411	+ 0.0689
B_3	0.0000	+ 0.3529	0.0000	+ 0.4502	- 0.989	+ 0.1516
B_4	- 0.2074	+ 0.2855	+ 0.1684	+ 0.2197	- 0.489	+ 0.0503
B_5	- 0.3357	+ 0.1091	+ 0.1041	+ 0.0865	- 0.769	+ 0.0433
B_6	- 0.3357	- 0.1091	- 0.1041	+ 0.0865	- 0.155	+ 0.0024
B_7	- 0.2074	- 0.2855	- 0.1684	- 0.2580	+ 0.414	- 0.0019
B_8	0.0000	- 0.3529	0.0000	- 1.0972	+ 0.473	- 0.0044
B_9	+ 0.2074	- 0.2855	+ 0.1684	- 0.3135	- 0.014	0.0000
B_{10}	+ 0.3357	- 0.1091	+ 0.1041	+ 0.0865	- 0.070	+ 0.0006
C_1	+ 0.2264	+ 0.0607	- 0.0152	+ 0.4017	- 0.525	+ 0.0486
C_2	+ 0.1658	+ 0.1658	- 0.0303	- 0.2372	- 0.565	+ 0.0776
C_3	+ 0.0607	+ 0.2264	- 0.0151	+ 0.2812	- 0.295	+ 0.0507
C_4	- 0.0607	+ 0.2264	+ 0.0151	+ 0.6945	- 0.558	+ 0.0566
C_5	- 0.1658	+ 0.1658	+ 0.0303	+ 1.5422	- 1.196	+ 0.1450
C_6	- 0.2264	+ 0.0607	+ 0.0151	+ 0.2637	- 0.385	+ 0.0420
C_7	- 0.2264	- 0.0607	- 0.0151	+ 0.3355	- 0.471	+ 0.0109
C_8	- 0.1658	- 0.1658	- 0.0303	+ 0.2177	- 0.456	+ 0.0319
C_9	- 0.0607	- 0.2264	- 0.0151	+ 0.3883	- 0.261	+ 0.0056
C_{10}	+ 0.0607	- 0.2264	+ 0.0151	+ 0.5937	- 0.150	+ 0.0013
C_{11}	+ 0.1658	- 0.1658	+ 0.0303	+ 0.3845	- 0.379	+ 0.0178
C_{12}	+ 0.2264	- 0.0607	+ 0.0151	+ 0.2905	- 0.166	+ 0.0007

TABLE IV.

Equations of transformation for the coefficients of the fourth order.

	a'_1	a'_2	a'_3	a'_4	a'_5	a'_6	a'_7	a'_8	a'_9	a'_{10}	a'_{11}	a'_{12}
A_1	+ 1.0000	0.0000		0.0000		0.0000				0.0000		
A_2	+ 1.0000	0.0000		0.0000		0.0000				0.0000		
B_1	+ 0.0091	+ 0.8183		- 0.2659		- 0.0280				+ 0.0863		
B_2	+ 0.4284	+ 0.1194		- 0.1643		- 0.3112				+ 0.2261		
B_3	+ 1.0000	0.0000		0.0000		0.0000				0.0000		
B_4	+ 0.4284	+ 0.1194		+ 0.1643		+ 0.3112				+ 0.2261		
B_5	+ 0.0091	+ 0.8183		+ 0.2659		+ 0.0280				+ 0.0863		
B_6	+ 0.0091	+ 0.8183		- 0.2659		- 0.0280				+ 0.0863		
B_7	+ 0.4284	+ 0.1194		- 0.1643		- 0.3112				+ 0.2261		
B_8	+ 1.0000	0.0000		0.0000		0.0000				0.0000		
B_9	+ 0.4284	+ 0.1194		+ 0.1643		+ 0.3112				+ 0.2261		
B_{10}	+ 0.0091	+ 0.8183		+ 0.2659		+ 0.0280				+ 0.0863		
C_1	+ 0.0045	+ 0.8705		- 0.2333		- 0.0152				+ 0.0625		
C_2	+ 0.2500	+ 0.2500		- 0.2500		- 0.2500				+ 0.2500		
C_3	+ 0.8705	+ 0.0045		- 0.0168		- 0.2333				+ 0.0625		
C_4	+ 0.8705	+ 0.0045		+ 0.0168		+ 0.2333				+ 0.0625		
C_5	+ 0.2500	+ 0.2500		+ 0.2500		+ 0.2500				+ 0.2500		
C_6	+ 0.0045	+ 0.8705		+ 0.2333		+ 0.0152				+ 0.0625		
C_7	+ 0.0045	+ 0.8705		- 0.2333		- 0.0152				+ 0.0625		
C_8	+ 0.2500	+ 0.2500		- 0.2500		- 0.2500				+ 0.2500		
C_9	+ 0.8705	+ 0.0045		- 0.0168		- 0.2333				+ 0.0625		
C_{10}	+ 0.8705	+ 0.0045		+ 0.0168		+ 0.2333				+ 0.0625		
C_{11}	+ 0.2500	+ 0.2500		+ 0.2500		+ 0.2500				+ 0.2500		
C_{12}	+ 0.0045	+ 0.8705		+ 0.2333		+ 0.0152				+ 0.0625		

	a'_{13}	a'_{14}	a'_{15}	S	a'_{16}	a'_{17}	a'_{18}	a'_{19}	a'_{20}
A_1				1.0000	+ 0.424	- 0.044	0.010	- 0.194	- 0.000
A_2				+ 1.0000	+ 0.145	+ 0.005	0.000	- 0.068	- 0.000
B_1				+ 0.6198	+ 1.988	- 0.659	+ 0.304	- 0.808	- 0.010
B_2				+ 0.2984	+ 0.483	- 0.168	+ 0.072	- 0.154	- 0.003
B_3				+ 1.0000	+ 0.075	- 0.000	+ 0.004	- 0.053	0.000
B_4				+ 1.2494	+ 0.419	- 0.026	+ 0.088	- 0.265	- 0.002
B_5				+ 1.2076	+ 2.050	- 0.767	+ 0.765	- 1.209	- 0.040
B_6				+ 0.6198	+ 0.778	- 0.081	+ 0.334	- 0.894	- 0.010
B_7				+ 0.2984	+ 3.144	- 1.163	+ 0.478	- 1.490	- 0.013
B_8				+ 1.0000	+ 0.810	- 0.092	+ 0.011	- 0.229	0.000
B_9				+ 1.2494	+ 0.593	- 0.163	+ 0.057	- 0.106	- 0.003
B_{10}				+ 1.2076	+ 1.254	- 0.126	+ 0.201	- 0.430	- 0.008
C_1				+ 0.6890	+ 3.062	+ 0.622	+ 0.512	- 1.708	- 0.013
C_2				+ 1.2500	+ 2.347	- 0.308	+ 0.442	- 1.051	- 0.016
C_3				+ 0.6874	+ 0.558	+ 0.022	+ 0.088	- 0.261	- 0.002
C_4				+ 1.1876	+ 0.673	+ 0.054	+ 0.019	- 0.192	0.000
C_5				+ 1.2500	+ 1.125	- 0.474	+ 0.296	- 0.548	- 0.013
C_6				+ 1.1860	+ 5.099	- 0.393	+ 0.822	- 2.739	- 0.021
C_7				+ 0.6890	+ 1.674	- 0.335	+ 0.450	- 1.044	- 0.016
C_8				+ 0.2500	+ 2.953	- 0.717	+ 0.687	- 1.846	- 0.020
C_9				+ 0.6874	+ 0.877	+ 0.013	+ 0.021	- 0.232	0.000
C_{10}				+ 1.1876	+ 0.419	- 0.087	+ 0.027	- 0.142	0.000
C_{11}				+ 1.2500	+ 1.857	- 0.262	+ 0.244	- 0.795	- 0.006
C_{12}				+ 1.1860	+ 1.844	- 0.943	+ 0.915	- 1.471	- 0.047

	b'_1	b'_2	b'_3	b'_4	b'_5	b'_6	b'_7
A_1	0.0000	0.9405	0.0009	0.0000	0.0000	0.0000	0.0000
A_2	0.0000	0.9405	0.0009	0.0000	0.0000	0.0000	0.0000
B_1	0.2060	0.0023	0.2483	+ 0.6071	— 0.2370	+ 0.0670	— 0.2159
B_2	0.0300	0.1078	0.2483	+ 0.0783	— 0.1465	+ 0.0413	— 0.0509
B_3	0.0000	0.2517	0.2483	0.0000	0.0000	0.0000	0.0000
B_4	0.0300	0.1078	0.2483	— 0.0783	+ 0.1465	— 0.0413	+ 0.0509
B_5	0.2060	0.0023	0.2483	— 0.0071	+ 0.2370	— 0.0670	+ 0.2159
B_6	0.0060	0.0023	0.2483	+ 0.0071	+ 0.2370	+ 0.0670	+ 0.2159
B_7	0.0300	0.1078	0.2483	+ 0.0783	+ 0.1465	+ 0.0413	+ 0.0509
B_8	0.0000	0.2517	0.2483	0.0000	0.0000	0.0000	0.0000
B_9	0.0300	0.1078	0.2483	— 0.0783	— 0.1465	— 0.0413	— 0.0509
B_{10}	0.2060	0.0023	0.2483	— 0.0071	— 0.2370	— 0.0670	— 0.2159
C_1	0.0034	0.0000	0.8791	+ 0.0001	— 0.2192	+ 0.0009	— 0.0136
C_2	0.0010	0.0010	0.8791	+ 0.0010	— 0.1605	+ 0.0010	— 0.0054
C_3	0.0000	0.0034	0.8791	+ 0.0009	— 0.0587	+ 0.0001	— 0.0003
C_4	0.0000	0.0034	0.8791	— 0.0009	+ 0.0587	— 0.0001	+ 0.0003
C_5	0.0010	0.0010	0.8791	— 0.0010	+ 0.1605	— 0.0010	+ 0.0054
C_6	0.0034	0.0000	0.8791	— 0.0001	+ 0.2192	— 0.0009	+ 0.0136
C_7	0.0034	0.0000	0.8791	+ 0.0001	+ 0.2192	+ 0.0009	+ 0.0136
C_8	0.0010	0.0010	0.8791	+ 0.0010	+ 0.1605	+ 0.0010	+ 0.0054
C_9	0.0000	0.0034	0.8791	+ 0.0009	+ 0.0587	+ 0.0001	+ 0.0003
C_{10}	0.0000	0.0034	0.8791	— 0.0009	— 0.0587	— 0.0001	— 0.0003
C_{11}	0.0010	0.0010	0.8791	— 0.0010	— 0.1605	— 0.0010	— 0.0054
C_{12}	0.0034	0.0000	0.8791	— 0.0001	— 0.2192	— 0.0009	— 0.0136

	b'_8	b'_9	b'_{10}	b'_{11}	b'_{12}	b'_{13}	b'_{14}
A_1	— 0.0051	— 0.1658	0.0000	0.0000	+ 0.0292	0.0000	0.0000
A_2	+ 0.0051	+ 0.1658	0.0000	0.0000	+ 0.0292	0.0000	0.0000
B_1	— 0.0770	— 0.0074	+ 0.0217	+ 0.2262	+ 0.0239	— 0.0701	— 0.0228
B_2	— 0.2016	— 0.1328	+ 0.0569	+ 0.0863	+ 0.1636	— 0.0701	— 0.0960
B_3	— 0.2492	— 0.2509	0.0000	0.0000	+ 0.2500	0.0000	0.0000
B_4	— 0.2016	— 0.1328	+ 0.0569	+ 0.0863	+ 0.1636	— 0.0701	+ 0.0960
B_5	— 0.0770	— 0.0074	+ 0.0217	+ 0.2262	+ 0.0239	— 0.0701	+ 0.0228
B_6	+ 0.0770	+ 0.0074	+ 0.0217	+ 0.2262	+ 0.0239	+ 0.0701	+ 0.0228
B_7	+ 0.2016	+ 0.1328	+ 0.0569	+ 0.0863	+ 0.1636	+ 0.0701	+ 0.0960
B_8	+ 0.2492	+ 0.2509	0.0000	0.0000	+ 0.2500	0.0000	0.0000
B_9	+ 0.2016	+ 0.1328	+ 0.0569	+ 0.0863	+ 0.1636	+ 0.0701	— 0.0960
B_{10}	+ 0.0770	+ 0.0074	+ 0.0217	+ 0.2262	+ 0.0239	+ 0.0701	— 0.0228
C_1	— 0.0587	— 0.0003	+ 0.0002	+ 0.0547	+ 0.0039	— 0.0037	— 0.0010
C_2	— 0.1605	— 0.0054	+ 0.0010	+ 0.0293	+ 0.0293	— 0.0054	— 0.0054
C_3	— 0.2192	— 0.0136	+ 0.0002	+ 0.0039	+ 0.0547	— 0.0010	— 0.0037
C_4	— 0.2192	— 0.0136	+ 0.0002	+ 0.0039	+ 0.0547	— 0.0010	+ 0.0037
C_5	— 0.1605	— 0.0054	+ 0.0010	+ 0.0293	+ 0.0293	— 0.0054	+ 0.0054
C_6	— 0.0587	— 0.0003	+ 0.0002	+ 0.0547	+ 0.0039	— 0.0037	+ 0.0010
C_7	+ 0.0587	+ 0.0003	+ 0.0002	+ 0.0547	+ 0.0039	+ 0.0037	+ 0.0010
C_8	+ 0.1605	+ 0.0054	+ 0.0010	+ 0.0293	+ 0.0293	+ 0.0054	+ 0.0054
C_9	+ 0.2192	+ 0.0136	+ 0.0002	+ 0.0039	+ 0.0547	+ 0.0010	+ 0.0037
C_{10}	+ 0.2192	+ 0.0136	+ 0.0002	+ 0.0039	+ 0.0547	+ 0.0010	— 0.0037
C_{11}	+ 0.1605	+ 0.0054	+ 0.0010	+ 0.0293	+ 0.0293	+ 0.0054	— 0.0054
C_{12}	+ 0.0587	+ 0.0003	+ 0.0002	+ 0.0547	+ 0.0039	+ 0.0037	— 0.0010

	b'_{15}	S	b'_{16}	b'_{17}	b'_{18}	b'_{19}	b'_{20}
A_1	0.0000	+ 0.7997	+ 1.698	— 0.272	+ 0.254	— 0.841	— 0.006
A_2	0.0000	+ 0.1415	+ 2.816	— 0.115	+ 0.577	— 1.154	— 0.024
B_1	+ 0.0735	+ 0.2458	+ 1.167	— 0.050	+ 0.092	— 0.158	— 0.005
B_2	+ 0.1189	+ 0.2335	+ 0.598	— 0.359	+ 0.153	— 0.152	— 0.013
B_3	0.0000	+ 0.2499	+ 1.065	— 0.436	+ 0.479	— 0.521	— 0.037
B_4	— 0.1189	+ 0.3433	+ 0.544	— 0.147	+ 0.164	— 0.266	— 0.008
B_5	— 0.0735	+ 0.9020	+ 2.084	— 0.580	+ 0.245	— 0.727	— 0.007
B_6	+ 0.0735	+ 1.5062	+ 0.347	— 0.023	+ 0.019	— 0.200	0.000
B_7	+ 0.1189	+ 1.6233	+ 3.981	— 0.058	+ 0.047	— 1.693	0.000
B_8	0.0000	+ 1.2501	+ 3.255	— 0.327	+ 0.071	— 1.258	0.000
B_9	— 0.1189	+ 0.5655	+ 1.142	+ 0.003	0.000	— 0.284	0.000
B_{10}	— 0.0735	+ 0.2596	+ 0.255	— 0.007	+ 0.005	— 0.105	0.000
C_1	+ 0.0147	+ 0.6605	+ 0.710	— 0.266	+ 0.174	— 0.313	— 0.008
C_2	+ 0.0293	+ 0.6294	+ 1.548	+ 0.134	+ 0.219	— 0.266	— 0.015
C_3	+ 0.0147	+ 0.6605	+ 0.314	— 0.199	+ 0.105	— 0.087	— 0.010
C_4	— 0.0147	+ 0.7545	+ 1.159	— 0.485	+ 0.195	— 0.319	— 0.010
C_5	— 0.0293	+ 0.9094	+ 2.693	— 1.482	+ 0.575	— 0.774	— 0.036
C_6	— 0.0147	+ 1.0767	+ 0.821	— 0.167	+ 0.122	— 0.186	— 0.007
C_7	+ 0.0147	+ 1.2535	+ 2.184	— 0.155	+ 0.095	— 0.681	— 0.001
C_8	+ 0.0293	+ 1.3146	+ 0.404	— 0.125	+ 0.132	— 0.314	— 0.005
C_9	+ 0.0147	+ 1.2535	+ 1.128	— 0.125	+ 0.042	— 0.327	0.000
C_{10}	— 0.0147	+ 1.0967	+ 1.150	— 0.119	+ 0.015	— 0.286	0.000
C_{11}	— 0.0293	+ 0.9024	+ 0.846	— 0.182	+ 0.090	— 0.320	— 0.002
C_{12}	— 0.0147	+ 0.7545	+ 1.150	— 0.048	+ 0.014	— 0.509	0.000

	c'_1	c'_2	c'_3	c'_4	c'_5	c'_6	c'_7	c'_8	c'_9	c'_{10}	c'_{11}
A_1	0.0600	0.0000		0.0000		+ 0.9848	- 0.1736		0.0000	0.0000	
A_2	0.0000	0.0000		0.0000		+ 0.9848	+ 0.1736		0.0000	0.0000	
B_1	+ 0.0796	- 0.7532		- 0.3960		- 0.1770	- 0.0208		+ 0.6073	+ 0.3368	
B_2	+ 0.8816	- 0.4656		+ 0.3959		- 0.1770	- 0.3738		+ 0.1433	- 0.2080	
B_3	0.0000	0.0000		0.0000		+ 0.7083	- 0.7059		0.0000	0.0000	
B_4	- 0.8816	+ 0.4656		+ 0.3959		- 0.1770	- 0.3738		- 0.1433	+ 0.2080	
B_5	- 0.0796	+ 0.7532		- 0.3960		- 0.1770	- 0.0208		- 0.6073	- 0.3368	
B_6	+ 0.0796	- 0.7532		- 0.3960		- 0.1770	+ 0.0208		- 0.6073	+ 0.3368	
B_7	+ 0.8816	- 0.4656		+ 0.3959		- 0.1770	+ 0.3738		- 0.1433	- 0.2080	
B_8	0.0000	0.0000		0.0000		+ 0.7083	+ 0.7059		0.0000	0.0000	
B_9	- 0.8816	+ 0.4656		+ 0.3959		- 0.1770	+ 0.3738		+ 0.1433	+ 0.2080	
B_{10}	- 0.0796	+ 0.7532		- 0.3960		- 0.1770	+ 0.0208		+ 0.6073	- 0.3368	
C_1	+ 0.0168	- 0.2332		- 0.1707		- 0.0458	- 0.0168		+ 0.8726	+ 0.1082	
C_2	+ 0.2500	- 0.2500		+ 0.1250		- 0.1250	- 0.3424		+ 0.3424	0.0000	
C_3	+ 0.2332	- 0.0168		+ 0.0458		+ 0.1707	- 0.8726		+ 0.0168	- 0.1082	
C_4	- 0.2332	+ 0.0168		+ 0.0458		+ 0.1707	- 0.8726		- 0.0168	+ 0.1082	
C_5	- 0.2500	+ 0.2500		+ 0.1250		- 0.1250	- 0.3424		- 0.3424	0.0000	
C_6	- 0.0168	+ 0.2332		- 0.1707		- 0.0458	- 0.0168		- 0.8726	- 0.1082	
C_7	+ 0.0168	- 0.2332		- 0.1707		- 0.0458	+ 0.0168		- 0.8726	+ 0.1082	
C_8	+ 0.2500	- 0.2500		+ 0.1250		- 0.1250	+ 0.3424		- 0.3424	0.0000	
C_9	+ 0.2332	- 0.0168		+ 0.0458		+ 0.1707	+ 0.8726		- 0.0168	- 0.1082	
C_{10}	- 0.2332	+ 0.0168		+ 0.0458		+ 0.1707	+ 0.8726		+ 0.0168	+ 0.1082	
C_{11}	- 0.2500	+ 0.2500		+ 0.1250		- 0.1250	+ 0.3424		+ 0.3424	0.0000	
C_{12}	- 0.0168	+ 0.2332		- 0.1707		- 0.0458	+ 0.0168		+ 0.8726	- 0.1082	

	c'_{12}	c'_{13}	c'_{14}	c'_{15}	S	c'_{16}	c'_{17}	c'_{18}	c'_{19}	c'_{20}
A_1		0.0000	0.0000		+ 0.8112	- 1.335	+ 0.244	- 0.078	+ 0.367	+ 0.001
A_2		0.0000	0.0000		+ 1.1584	+ 0.108	+ 0.092	+ 0.006	- 0.100	- 0.000
B_1		+ 0.0640	- 0.1973		- 0.4566	- 0.576	+ 0.936	- 0.546	+ 0.209	+ 0.032
B_2		+ 0.2716	- 0.1973		+ 0.2707	- 0.833	+ 0.579	- 0.287	+ 0.329	+ 0.016
B_3		0.0000	0.0000		+ 0.0024	+ 0.110	+ 0.041	- 0.050	- 0.021	+ 0.001
B_4		- 0.2716	- 0.1973		- 0.9751	+ 0.534	- 0.003	+ 0.312	- 0.433	- 0.013
B_5		- 0.0640	- 0.1973		- 1.1256	+ 0.140	- 0.655	+ 1.099	- 0.363	- 0.104
B_6		- 0.0640	+ 0.1973		- 1.3630	- 0.707	+ 0.155	+ 0.088	+ 0.876	- 0.015
B_7		- 0.2716	+ 0.1973		+ 0.5831	- 5.741	+ 1.970	- 0.734	+ 2.777	+ 0.016
B_8		0.0000	0.0000		+ 1.4142	+ 0.597	- 0.197	+ 0.044	- 0.309	- 0.000
B_9		+ 0.2716	+ 0.1973		+ 0.9969	+ 1.937	- 0.437	+ 0.073	- 0.289	+ 0.000
B_{10}		+ 0.0640	+ 0.1973		+ 0.6532	+ 1.018	+ 0.326	+ 0.040	- 0.570	+ 0.007
C_1		+ 0.0627	- 0.2338		+ 0.3600	+ 2.904	+ 0.652	- 0.865	- 0.309	+ 0.046
C_2		+ 0.3424	- 0.3424		0.0000	+ 3.272	+ 0.463	- 0.760	- 0.518	+ 0.082
C_3		+ 0.2338	- 0.0627		- 0.3600	+ 0.430	+ 0.139	- 0.235	- 0.099	+ 0.014
C_4		- 0.2338	- 0.0627		- 1.0776	- 0.473	+ 0.062	+ 0.103	+ 0.098	- 0.002
C_5		- 0.3424	- 0.3424		- 1.3696	- 0.679	- 0.932	+ 0.740	+ 0.060	- 0.068
C_6		- 0.0627	- 0.2338		- 1.2942	- 4.019	- 2.039	+ 1.083	+ 1.053	- 0.062
C_7		- 0.0627	+ 0.2338		- 1.0094	- 1.998	- 0.433	+ 0.225	+ 1.088	- 0.033
C_8		- 0.3424	+ 0.3424		0.0000	- 0.351	- 1.164	+ 0.979	- 0.180	- 0.056
C_9		- 0.2338	+ 0.0627		+ 1.0094	- 2.013	- 0.167	+ 0.041	+ 0.312	- 0.001
C_{10}		+ 0.2338	+ 0.0627		+ 1.2942	- 0.494	+ 0.227	- 0.041	+ 0.081	+ 0.001
C_{11}		+ 0.3424	+ 0.3424		+ 1.3696	- 1.573	+ 0.449	- 0.354	- 0.118	+ 0.019
C_{12}		+ 0.0627	+ 0.2338		+ 1.0776	+ 1.115	- 0.909	- 0.218	- 0.232	+ 0.030

	d'_1	d'_2	d'_3	d'_4	d'_5	d'_6	d'_7	d'_8	d'_9	d'_{10}	d'_{11}
A_1	0.0000	0.0000		+ 0.9551	— 0.0052	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
A_2	0.0000	0.0000		+ 0.9551	+ 0.0052	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_1	+ 0.3780	— 0.0400		— 0.0889	— 0.1087	— 0.1986	— 0.2970	+ 0.3345	+ 0.0906	— 0.1690	+ 0.2074
B_2	+ 0.2336	— 0.4424		— 0.0889	— 0.2346	+ 0.1986	— 0.2970	+ 0.2067	+ 0.4086	+ 0.1044	+ 0.3356
B_3	0.0000	0.0000		+ 0.3554	— 0.3517	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_4	— 0.2336	+ 0.4424		— 0.0889	— 0.2846	— 0.1986	— 0.2970	— 0.2067	— 0.4086	— 0.1044	— 0.3356
B_5	— 0.3780	+ 0.0400		— 0.0889	— 0.1087	— 0.1986	— 0.2970	— 0.3345	— 0.0906	+ 0.1690	— 0.2074
B_6	+ 0.3780	— 0.0400		— 0.0889	+ 0.1087	— 0.1986	+ 0.2970	— 0.3345	— 0.0906	— 0.1690	+ 0.2074
B_7	+ 0.2336	— 0.4424		— 0.0889	+ 0.2846	+ 0.1986	+ 0.2970	— 0.2067	— 0.4086	+ 0.1044	+ 0.3356
B_8	0.0000	0.0000		+ 0.3554	+ 0.3517	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
B_9	— 0.2336	+ 0.4424		— 0.0889	+ 0.2846	+ 0.1986	+ 0.2970	+ 0.2067	+ 0.4086	— 0.1044	— 0.3356
B_{10}	— 0.3780	+ 0.0400		— 0.0889	+ 0.1087	— 0.1986	+ 0.2970	+ 0.3345	+ 0.0906	+ 0.1690	— 0.2074
C_1	+ 0.0144	— 0.0012		— 0.0029	— 0.2340	— 0.0106	— 0.0438	+ 0.8769	+ 0.0117	— 0.0066	+ 0.1172
C_2	+ 0.0156	— 0.0156		— 0.0078	— 0.6420	+ 0.0078	— 0.0642	+ 0.6420	+ 0.0642	0.0000	+ 0.2344
C_3	+ 0.0012	— 0.0144		+ 0.0106	— 0.8769	+ 0.0029	— 0.0117	+ 0.2350	— 0.0438	+ 0.0066	+ 0.1172
C_4	— 0.0012	+ 0.0144		+ 0.0106	— 0.8769	+ 0.0029	— 0.0117	— 0.2350	— 0.0438	— 0.0066	+ 0.1172
C_5	— 0.0156	+ 0.0156		— 0.0078	— 0.6420	+ 0.0078	— 0.0642	— 0.6420	— 0.0642	0.0000	— 0.2344
C_6	— 0.0144	+ 0.0012		— 0.0029	— 0.2340	— 0.0106	— 0.0438	— 0.8769	— 0.0117	+ 0.0066	— 0.1172
C_7	+ 0.0144	— 0.0012		— 0.0029	+ 0.2340	— 0.0106	+ 0.0438	+ 0.8769	+ 0.0117	— 0.0066	+ 0.1172
C_8	+ 0.0156	— 0.0156		— 0.0078	+ 0.6420	+ 0.0078	+ 0.0642	— 0.6420	— 0.0642	0.0000	+ 0.2344
C_9	+ 0.0012	— 0.0144		+ 0.0106	— 0.8769	+ 0.0029	+ 0.0117	— 0.2350	— 0.0438	+ 0.0066	+ 0.1172
C_{10}	— 0.0012	+ 0.0144		+ 0.0106	— 0.8769	+ 0.0029	+ 0.0117	+ 0.2350	— 0.0438	— 0.0066	— 0.1172
C_{11}	— 0.0156	+ 0.0156		— 0.0078	+ 0.6420	+ 0.0078	+ 0.0642	+ 0.6420	+ 0.0642	0.0000	— 0.2344
C_{12}	— 0.0144	+ 0.0012		— 0.0029	+ 0.2340	— 0.0106	+ 0.0438	+ 0.8769	+ 0.0117	+ 0.0066	— 0.1172

	d'_{12}	d'_{13}	d'_{14}	d'_{15}	S	d'_{16}	d'_{17}	d'_{18}	d'_{19}	d'_{20}
A_1	0.0000	0.0000	— 0.1685	+ 0.0296	+ 0.8110	— 3.195	+ 0.573	— 0.261	+ 0.764	+ 0.007
A_2	0.0000	0.0000	+ 0.1685	+ 0.0296	+ 1.1584	— 0.009	+ 0.215	+ 0.164	— 0.410	+ 0.005
B_1	— 0.2074	+ 0.2404	+ 0.1876	— 0.2856	+ 0.0492	— 0.248	— 0.433	— 0.251	+ 0.092	+ 0.022
B_2	— 0.3356	— 0.2016	+ 0.0105	+ 0.1091	— 0.0430	— 1.609	+ 0.971	— 0.373	+ 0.326	+ 0.035
B_3	0.0000	0.0000	— 0.3541	+ 0.3529	+ 0.0025	+ 0.780	— 0.170	— 0.129	— 0.067	+ 0.024
B_4	+ 0.3356	+ 0.2016	+ 0.0105	+ 0.1091	— 0.0616	+ 1.059	— 0.194	+ 0.374	— 0.433	— 0.025
B_5	+ 0.2074	— 0.2404	+ 0.1876	— 0.2856	— 1.6317	+ 0.555	— 1.351	+ 0.811	— 0.281	— 0.043
B_6	— 0.2074	— 0.2404	— 0.1876	+ 0.2856	— 0.8575	— 0.643	+ 0.157	+ 0.090	+ 0.415	— 0.002
B_7	— 0.3356	+ 0.2016	— 0.0105	+ 0.1091	+ 0.2718	— 7.220	+ 0.680	— 0.349	+ 2.961	+ 0.002
B_8	0.0000	0.0000	+ 0.3541	+ 0.3529	+ 1.4142	+ 1.857	— 0.309	+ 0.106	— 0.725	— 0.001
B_9	+ 0.3356	— 0.2016	— 0.0105	+ 0.1091	+ 1.3080	+ 2.207	— 0.184	— 0.007	— 0.473	0.000
B_{10}	+ 0.2074	+ 0.2404	— 0.1876	— 0.2856	+ 0.1475	+ 0.484	+ 0.053	— 0.040	— 0.282	0.000
C_1	— 0.1172	+ 0.0467	+ 0.0281	— 0.2030	+ 0.4748	+ 0.923	+ 0.064	— 0.354	— 0.132	+ 0.036
C_2	— 0.2344	— 0.0214	+ 0.0214	0.0000	0.0000	+ 4.046	+ 0.465	— 0.334	— 0.261	+ 0.061
C_3	— 0.1172	— 0.0281	— 0.0467	+ 0.2030	— 0.4747	+ 0.031	+ 0.003	— 0.111	— 0.057	+ 0.029
C_4	+ 0.1172	+ 0.0281	— 0.0467	+ 0.2030	— 0.9629	— 1.230	+ 1.405	+ 0.101	+ 0.126	— 0.014
C_5	+ 0.2344	+ 0.0214	+ 0.0214	0.0000	— 1.3696	— 0.231	— 1.258	+ 0.872	+ 0.071	— 0.111
C_6	+ 0.1172	— 0.0467	+ 0.0281	— 0.2030	— 1.4090	— 0.635	— 0.009	+ 0.233	+ 0.274	— 0.035
C_7	— 0.1172	— 0.0467	— 0.0281	— 0.2030	— 0.8946	— 1.680	— 0.228	+ 0.310	+ 0.879	— 0.009
C_8	— 0.2344	+ 0.0214	— 0.0214	0.0000	0.0000	+ 0.143	— 0.369	+ 0.395	— 0.072	— 0.027
C_9	— 0.1172	+ 0.0281	+ 0.0467	+ 0.2030	+ 0.8945	— 2.669	+ 0.272	+ 0.041	+ 0.371	— 0.001
C_{10}	+ 0.1172	— 0.0281	+ 0.0467	+ 0.2030	+ 1.4091	— 1.009	+ 0.297	— 0.051	+ 0.114	0.000
C_{11}	+ 0.2344	— 0.0214	— 0.0214	0.0000	+ 1.3696	— 1.099	+ 0.199	— 0.226	— 0.075	+ 0.011
C_{12}	+ 0.1172	+ 0.0467	— 0.0281	— 0.2030	+ 0.9598	+ 0.415	+ 0.307	— 0.169	— 0.136	+ 0.001

	e'_1	e'_2	e'_3	e'_4	e'_5	e'_6	e'_7	e'_8	e'_9	e'_{10}	e'_{11}	e'_{12}
A_1	0.0030	0.0000		0.0000		0.0000	0.0000		0.0000	+ 0.9698	+ 0.0301	0.0000
A_2	0.0000	0.0000		0.0000		0.0000	0.0000		0.0000	+ 0.9698	+ 0.0301	0.0000
B_1	+ 0.2598	+ 0.2598		+ 0.3580		- 0.3580	- 0.1361		- 0.4193	+ 0.2420	+ 0.0475	+ 0.4507
B_2	+ 0.6804	+ 0.6804		- 0.2210		+ 0.2212	- 0.5070		- 0.4193	- 0.1788	+ 0.3261	+ 0.1722
B_3	0.0000	0.0000		0.0000		0.0000	0.0000		0.0000	+ 0.5017	+ 0.4983	0.0000
B_4	+ 0.6804	+ 0.6804		+ 0.2210		- 0.2212	+ 0.5070		- 0.4193	- 0.1788	+ 0.3261	+ 0.1722
B_5	+ 0.2598	+ 0.2598		- 0.3580		+ 0.3580	+ 0.1361		- 0.4193	+ 0.2420	+ 0.0475	+ 0.4507
B_6	+ 0.2598	+ 0.2598		+ 0.3580		- 0.3580	+ 0.1361		+ 0.4193	+ 0.2420	+ 0.0475	+ 0.4507
B_7	+ 0.6804	+ 0.6804		- 0.2210		+ 0.2212	+ 0.5070		+ 0.4193	- 0.1788	+ 0.3261	+ 0.1722
B_8	0.0000	0.0000		0.0000		0.0000	0.0000		0.0000	+ 0.5017	+ 0.4983	0.0000
B_9	+ 0.6804	+ 0.6804		+ 0.2210		- 0.2212	- 0.5070		+ 0.4193	- 0.1788	+ 0.3261	+ 0.1722
B_{10}	+ 0.2598	+ 0.2598		- 0.3580		+ 0.3580	- 0.1361		+ 0.4193	+ 0.2420	+ 0.0475	+ 0.4507
C_1	+ 0.0234	+ 0.0234		+ 0.0405		- 0.0406	- 0.0469		- 0.1751	+ 0.0391	+ 0.0628	+ 0.8748
C_2	+ 0.0942	+ 0.0942		0.0000		0.0000	- 0.2568		- 0.2568	- 0.0312	+ 0.4688	+ 0.4688
C_3	+ 0.0234	+ 0.0234		- 0.0406		+ 0.0406	- 0.1751		- 0.0469	+ 0.0391	+ 0.8748	+ 0.0628
C_4	+ 0.0234	+ 0.0234		+ 0.0406		- 0.0406	+ 0.1751		- 0.0469	+ 0.0391	+ 0.8748	+ 0.0628
C_5	+ 0.0942	+ 0.0942		0.0000		0.0000	+ 0.2568		- 0.2568	- 0.0312	+ 0.4688	+ 0.4688
C_6	+ 0.0234	+ 0.0234		- 0.0405		+ 0.0406	+ 0.0469		- 0.1751	+ 0.0391	+ 0.0628	+ 0.8748
C_7	+ 0.0234	+ 0.0234		+ 0.0405		- 0.0406	+ 0.0469		+ 0.1751	+ 0.0391	+ 0.0628	+ 0.8748
C_8	+ 0.0942	+ 0.0942		0.0000		0.0000	+ 0.2568		+ 0.2568	- 0.0312	+ 0.4688	+ 0.4688
C_9	+ 0.0234	+ 0.0234		- 0.0406		+ 0.0405	+ 0.1751		+ 0.0469	+ 0.0391	+ 0.8748	+ 0.0628
C_{10}	+ 0.0234	+ 0.0234		+ 0.0406		- 0.0405	- 0.1751		+ 0.0469	+ 0.0391	+ 0.8748	+ 0.0628
C_{11}	+ 0.0942	+ 0.0942		0.0000		0.0000	- 0.2568		+ 0.2568	- 0.0312	+ 0.4688	+ 0.4688
C_{12}	+ 0.0234	+ 0.0234		- 0.0405		+ 0.0406	- 0.0469		+ 0.1751	+ 0.0391	+ 0.0628	+ 0.8748

	e'_{13}	e'_{14}	e'_{15}	S	e'_{16}	e'_{17}	e'_{18}	e'_{19}	e'_{20}
A_1	- 0.1710	0.0000	0.0000	+ 0.8289	+ 3.376	- 0.466	+ 0.209	- 0.894	- 0.003
A_2	+ 0.1710	0.0000	0.0000	+ 1.1709	+ 0.809	+ 0.102	+ 0.153	- 0.580	- 0.001
B_1	+ 0.2648	- 0.3393	- 0.1464	+ 0.4835	+ 3.196	+ 0.277	+ 0.408	- 0.721	- 0.039
B_2	+ 0.0148	+ 0.2831	- 0.2369	+ 0.7452	+ 1.611	- 0.984	+ 0.450	- 0.394	- 0.036
B_3	- 0.5000	0.0000	0.0000	+ 0.5000	+ 0.717	- 0.206	+ 0.156	- 0.333	- 0.006
B_4	+ 0.0148	- 0.2831	+ 0.2369	+ 1.8063	+ 0.240	- 0.018	+ 0.448	- 0.619	- 0.027
B_5	+ 0.2648	+ 0.3393	+ 0.1464	+ 1.7271	+ 2.478	- 0.803	+ 1.057	- 1.888	- 0.100
B_6	- 0.2648	+ 0.3393	- 0.1464	+ 1.7433	+ 1.116	- 0.204	+ 0.085	- 0.953	- 0.008
B_7	- 0.0148	- 0.2831	- 0.2369	+ 2.1420	+ 10.392	- 2.103	+ 0.823	- 3.823	- 0.007
B_8	+ 0.5000	0.0000	0.0000	+ 1.5000	+ 1.949	- 0.141	+ 0.081	- 1.125	- 0.001
B_9	- 0.0148	+ 0.2831	+ 0.2369	+ 2.0276	+ 3.594	- 0.563	+ 0.087	- 0.456	0.000
B_{10}	- 0.2648	- 0.3393	+ 0.1464	+ 0.0853	+ 1.448	- 0.167	+ 0.048	- 0.520	- 0.003
C_1	+ 0.1128	- 0.1869	- 0.2344	+ 0.4929	+ 3.831	- 0.811	+ 0.544	- 1.470	- 0.061
C_2	+ 0.0856	+ 0.0856	- 0.4688	+ 0.2836	+ 7.169	- 0.812	+ 0.442	- 1.089	- 0.092
C_3	- 0.1869	+ 0.1128	- 0.2344	+ 0.4930	+ 0.687	- 0.153	+ 0.185	- 0.306	- 0.030
C_4	- 0.1866	- 0.1128	+ 0.2344	+ 1.0864	+ 3.397	- 0.045	+ 0.149	- 0.502	- 0.008
C_5	+ 0.0856	- 0.0856	+ 0.4688	+ 1.5636	+ 5.098	- 0.494	+ 0.795	- 1.303	- 0.131
C_6	+ 0.1128	+ 0.1869	+ 0.2344	+ 1.4295	+ 7.360	- 1.066	+ 0.442	- 1.476	- 0.070
C_7	- 0.1128	+ 0.1869	- 0.2344	+ 1.0851	+ 3.783	+ 0.016	+ 0.242	- 1.829	- 0.025
C_8	- 0.0856	- 0.0856	- 0.4688	+ 0.9684	+ 1.965	- 0.733	+ 0.647	- 1.556	- 0.058
C_9	+ 0.1829	- 0.1128	- 0.2344	+ 1.0851	+ 4.306	+ 0.048	+ 0.024	- 0.603	- 0.002
C_{10}	+ 0.1869	+ 0.1128	+ 0.2344	+ 1.4295	+ 1.700	- 0.280	+ 0.059	- 0.409	- 0.001
C_{11}	- 0.0856	+ 0.0856	+ 0.0688	+ 1.5636	+ 3.962	- 0.138	+ 0.270	- 1.012	- 0.022
C_{12}	- 0.1128	- 0.1869	+ 0.2344	+ 1.0865	+ 2.183	+ 0.029	+ 0.539	- 1.735	- 0.007

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